Dr.Babasaheb Ambedkar Open University

DOR

DIPLOMA IN OPERATION RESEARCH

Block

2

Assignment and Transportation Problems

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Unit : 3 : Assignment Problem

Introduction

The problem of assignment is related with the assigning various duties to the various class of the people so that the objectives are optimized. The problem of assignment arises in the case when various alternatives are available for the work distribution. The assignment problem is a special type of the problem in which the main objective is to assign a number of origins or the sources to the equal number of destination at a minimum expense or maximum profit or some other related objectives.

The problem of assignment arises because avaibility of more than one sources for work distribution or available resources such as men, machines, etc. have varying degrees 18of efficiency for performing different activities for satisfying the given objectives. Therefore, expense, profit or time of performing the different activities may be different. Thus, the problem is to find how should the assignments be made among the various people so that objectives are optimized.

Some of the problems where the assignment technique may be useful are: Assignment of workers to machines, salesmen to different sales areas, clerks to various checkout counters, teåchers to class etc.

Structure of the chapter:

3.1 Objectives:

By the end of this chapter the student will learn to know about

What is assignment problems
How such problems be solved

3.2 Mathematical Statement Of The Problem

Given n resources (or facilities or available alternatives for the distribution) and n activities (or jobs), and effectiveness or the objectives (in terms of expense, profit, time or any other objectives), of each resource (facility) for each activity (job), the problem lies in assigning each resource to one and only one activity (job) so that the given measure of effectiveness or objective is, optimized. The data matrix for this problem is shown below in the following table.

Resources	Activities J_1 J_2 ....J_n	Supply
W_1	C_{11} C_{12}... C_{1n}	1
W_2	C_{21} C_{22} C_{2n}	1
.		
.		
W_n	C_{n1} C_{n2}.... C_{nn}	1
Demand	1 1.... 1	N

From the above table, it can be noted that the data matrix is the same as the transportation expense matrix in every respect with the only difference that supply (or availability) of each of the resources and the demand at each of the destinations is taken to be one which is not there in transportation matrix. It is due to only this fact that assignments are made on a one-to-one basis and in this manner it differs from the transportation matrix.

Let x_{ij} denote the assignment of facility i to job j such that

$$x_{ij} = \begin{cases} 1 \text{ if facility i is assigned to job j} \\ 0 \text{ otherwise} \end{cases}$$

Then, the mathematical model of the assignment problem can be stated as:

$$\text{Minimize } Z = \sum_{I=1}^{u} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \text{ for all i (resource availability)}$$

$$\sum_{I=1}^{n} x_{ij} = 1; \text{ for all j (activity requirement)}$$

and x_{ij}, = 0 or 1, for all i and j

where. c_{ij} represents the expense of assignment of resource i to activity j.

From the above discussion, it is clear that though the assignment problem is similar to the transportation problem there is some variation from the transportation problem with following characteristics: (i) the expense matrix is always square matrix, and (ii) the solution for the problem or the result of the assignment problem would always be such that there would be only one assignment in a given row or column of the expense matrix.

It is important to note that, in an assignment problem if a constant is added to or subtracted from every element of any row or column in the given expense matrix, an assignment that minimizes the total expense in one matrix can also minimizes the total expense in the other matrix.

3.3 Solution Method Of Assignment Problem

Step 1. Check whether the number of rows and columns in the cost matrix are equal. If not, add dummy rows (columns) to form a square matrix.

Step 2. In the square cost matrix

(i) Reduce each row element by the lowest element of that row

(ii) Repeat the above for each column.

Step 3. In the reduced matrix, search for optimum solution as follows :

(i) Examine the rows successively until a row with exactly single zero is found. Mark this zero by enrectangling () and cross out () all other zeros of the corresponding column. Proceed in this manner until all rows have been examined. In case of at least two zeros in a particular row and column, choose arbitrarily any one of these and cross out all other zeros of that row and column.

(ii) Repeat the process for columns.

(iii) If each row and each column has one and only one marked zero, the optimum allocation is attained which is indicated by the marked positions. Otherwise go to next step.

Step 4. Draw the minimum number of lines passing through all the zeros as follows :

(i) Tick () rows that do not have assignments.

(ii) Tick () columns that have zeros in ticked rows.

(iii) Tick () rows that have assignment in ticked columns.

(iv) Repeat (ii) and (iii) until the chain is completed.

(v) Draw straight lines through all unticked rows and ticked columns.

Step 5. If the minimum number of lines passing through all the zeros is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crossed in step 3. Otherwise go to next step.

Step 6. Revise the costs matrix as follows :

(i) Find the smallest element not covered by any of the lines of step 4.

(ii) Subtract this from all the uncrossed elements and add the same at the point of intersection of the two lines.

(iii) Other elements crossed by the lines remain unchanged.

Step 7. Go to step 4 and repeat the procedure till an optimum solution is attained.

3.4 Variations Of The Assignment Problem

Multiple Optimal Solutions

When there are more than one objective of the operation or the activity then it is treaed as multi objective problem. Here, we have to take care of more than one objectives in the mind. While finding optimum solution of the problem we have to take care of more than one objectives. Multi-objective optimization deals with solving optimization problems, which involve multiple objectives. In real life almost all the problems involve multi objective problems and they are not having in fact single solution. Most real-world search and optimization problems involve multiple objectives and should be ideally formulated and solved as a multi-objective optimization problem. However, the task of multi-objective optimization is different from that of single-objective optimization in that in multi-objective optimization, there is usually no single solution, which is optimum with respect to all objectives. So, more than one solution is available for the problem. Such solutions are known as Pareto optimal solutions. Here, one of the goals of multi-objective optimization must have to find as many Pareto-optimal solutions as possible.

Maximization Case in Assignment Problem

The objectives in case of cost allocation problem may be related with the minimization but if it is related with the profit then it can be denoted as maximization problem. There may be situations when the assignment problem calls for maximization of profit, revenue, etc., as the objective function. Such problems may be solved by converting the given maximization problem into a minimization problem in either of the following two ways:

(i) Put a negative sign before each of the payoff elements in the assignment table so as to convert the profit values into expense values. Or

(ii) Locate the largest payoff element in the assignment table and then subtract all the element of the table from the largest element.

The transformed assignment problem so obtained can be solved by using the Hungarian method.

Unbalanced Assignment Problem

For solving the problem of assignment, it must have a square matrix. The Hungarian method of assignment discussed above requires that the number of columns and rows in the assignment matrix be equal. However, when the given expense matrix is not a square matrix, the assignment problem is called an unbalanced problem. In such cases a dummy row(s) or column(s) are added in the matrix (with zeros as the expense elements) to make it a square matrix. After making the given expense matrix a square matrix, the Hungarian method may be used to solve the problem.

Restrictions on Assignments

Sometimes it may happen that a particular person or a particular machine is prohibited from doing any specific operations, before we start solving the problem of assignment. Such type of problem is known as restricted assignment problem. Sometimes it may happen that a particular resource cannot be assigned to perform a particular activity. In such cases, the expense of performing that particular activity by a particular resource is considered to be very large (written as M or ∞) so as to prohibit the entry of this pair of resource-activity into the final solution.

Travelling Salesman Problem

This problem is mainly related with the solving of problems of traveling sales person. Here, emphasis is on reducing the expenses of the traveling sales person as well as making a route in a such a way so that the sales person is coving the total distance. The traveling salesman problem may be solved as an assignment problem with two additional conditions on the choice of assignment. That is, how should a traveling salesman travel starting from his home city, visiting each city only once and returning to his home city so that the total distance is minimized. For example, given n cities and distances d_{ij} (expense c_{ij} or time t_{ij}) from city I to city j, the salesman starts from city 1, then any permutation of 2, 3,..., n represents the number of possible ways for his tour. Thus, there are (n - 1)! possible ways of his tour. Now the problem is to select an optimal route that could achieve the objective of the salesman. To formulate and solve this problem, let us define:

$$x_{ij} = \begin{cases} 1, \text{ if salesman travels from city i to city j} \\ 0, \text{ otherwise} \end{cases}$$

Since each city can be visited only once, we have

$$\sum_{I=1}^{n-1} X_{ij} = 1,\ j = 1,2,...,n;\ i \neq j$$

Again, since the salesman has to leave each city except city n, we have

$$\sum^{n} X_{ij} = 1,\ i = 1,2,...,n;\ i \neq j$$

The objective function is then

$$\text{Minimize } Z = \sum_{I=1}^{n-1} \sum_{j=1}^{n} d_{ij}\, x_{ij}$$

Illustration:

A department has five employees with five jobs to be performed. The time (in hours) each men will take to perform each job is given in the effectiveness matrix.

		Employees				
		I	*II*	*III*	*IV*	*V*
Jobs	*A*	10	5	13	15	16
	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

Solution:

Applying Step 2 of the algorithm, we get the reduced opportunity time matrix as shown in Table.

	I	*II*	*III*	*IV*	*V*
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

Steps 3 and 4 (a) We examine all the rows starting from A one-by-one until a row containing only single zero element is located. Here rows A, B and E have only one zero element in the cells (A, II), (B, I) and (E, IV). Assignment is made in these cells. All zeros in the assigned columns are now crossed off as shown in Table.

(b) We now examine each column starting from column I. There is one zero in column III, cell (C, III). Assignment is made in this cell. Thus cell (C, V) is crossed off. All zeros in the table now are either assigned or crossed off as shown in Table.

The solution is not optimal because only four assignments are made.

Step 5 Cover the zeros with minimum number of lines (= 4) as explained below:

(a) Mark () row D since it has no assignment. Then

(b) Mark () columns I and IV since row D has zero element in these columns.

(c) Mark () rows B and E since columns I and IV have an assignment in rows B and E, respectively.

(d) Since no other rows or columns can be marked, draw straight lines through the unmarked rows A and C and the marked columns I and IV, as shown in Table.

	I	*II*	*III*	*IV*	*V*
A	5	$\boxed{0}$	8	10	11
B	$\boxed{0}$	6	15	10	3
C	8	5	$\boxed{0}$	~~0~~	~~0~~
D	~~0~~	4	2	~~0~~	5
E	3	5	6	$\boxed{0}$	8

	I	II	III	IV	V
A	7	0	8	12	11
B	0	4	13	10	1
C	10	5	0	2	0
D	0	2	0	0	3
E	3	3	4	0	6

Step 6 Develop the new revised table by selecting the smallest element among all uncovered elements by the lines in Table; viz., 2. Subtract k=2 from uncovered elements including itself and add it to elements 5,10, 8 and 0 in cells (A, I), (A, IV), (C, I) and (C, IV), respectively which lie at the intersection of two lines. Another, revised table so obtained is shown in Table.

Step 7 Repeat Steps 3 to 6 to find a new solution. The new assignment is shown in Table.

Since the number of assignments (= 5) equals the number of rows (or columns), the solution is optimal. The pattern of assignments among jobs and employees with their respective time (in hours) is given below:

Job	Employee	Time (in hours)
A	II	5
B	I	3
C	V	2
D	III	9
E	IV	4
		Total 23

Exercise:

1. Suchita Ltd. has four territories open, and four salesmen available for an assignment. The territories are not equally rich in their sales potential. It is estimated that a typical salesman operating in each territory would bring in the following annual sales:

Territory :	I	II	III	IV
Annual sales (Rs) :	1,20,000	1,05,000	84,000	63,000

The four salesmen also differ in their ability. It is estimated that, working under the same conditions, their yearly sales would be proportionately as follows:

Salesmen :	A	B	C	D
Proportion :	7	5	5	4

If the criterion is maximum expected total sales, the intuitive answer is to assign the best salesman to the richest territory, the next best salesman to the second richest, and so on; verify this answer by the assignment technique.

Solution:

To avoid the fractional values of annual sales of each salesman in each territory, for convenience consider their yearly sales as 21 (i.e. the sum of sales proportions), taking Rs 1,000 as one unit. Now divide the individual sales in each territory by 21 to obtain the required annual sales by each salesman. The maximum sales matrix so obtained is given in Table.

Table Effectiveness Matrix

Converting Maximization Problem into Minimization Problem The given maximization assignment problem can be converted into a minimization assignment problem by subtracting from the highest element (i.e. 42), all the elements of the given table. The new data so obtained is given in Table.

Table Equivalent Expense Data

	I	II	III	IV
A .	0	7	14	21
B	12	17	22	27
C	12	17	22	27
D	18	22	26	30

	I	II	///	IV
A	0	3	6	9
B	0	1	2	3
C	0	1	2	3
D	0	0	0	0

Table 10.20

	I	II	III	IV	
A	$\boxed{0}$	3	6	9	✓
B	$\cancel{0}$	1	2	3	✓
C	$\cancel{0}$	1	2	3	✓
D	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
	✓				

The solution shown in is not optimal since only three assignments are made. Cover the zeros with the minimum number of lines (= 2) as shown in Table by the usual method discussed earlier.

Develop the revised expense matrix by selecting the minimum element (= 1) among all uncovered elements by the lines. Subtract 1 from each uncovered element including itself and add it to the element at the intersection of two lines. A revised expense table so obtained is shown in Table.

	I	II	III	IV
A	$\boxed{0}$	2	5	8
B	$\cancel{0}$	$\boxed{0}$	1	2
C	$\cancel{0}$	$\cancel{0}$	1	2
D	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Repeat Steps 1 to 3 to mark the assignments in Table as per guidelines discussed earlier. Two alternative optimal assignments are shown in Tables(a) and (b).

Table (a) Table (b)

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
C	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

	I	II	III	IV
A	$\boxed{0}$	2	4	7
B	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
C	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
D	2	1	$\cancel{0}$	$\boxed{0}$

The pattern of two alternative optimal assignments among territories and salesmen with their respective sales volume (in Rs 1,000) is given in the table.

Assignment Set I		
Salesman	Territory	Sales(Rs)
A	I	42
B	III	20
C	II	25
D	IV	12
	Total	Rs 99
Assignment Set II		
Salesman	Territory	Sales(Rs)
A	I	42
B	II	25
C	III	20

D	IV	12
	Total	Rs 99

2. In Sanket Ltd., in the modification of a plant layout of a factory four new machines M_1, M_2, M_3, and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The expense of locating a machine at a place (in hundred rupees) is as follows.

		Location				
		A	*B*	*C*	*D*	*E*
Machine	M_1	9	11	15	10	11
	M_2	12	9	—	10	9
	M_3	—	11	14	11	7
	M_4	14	8	12	7	8

Find the optimal assignment schedule.

Solution -As the expense matrix is not balanced, add one dummy row (machine) with a zero expense element in that row. Also assign a high expense, denoted by M, to the pair (M_2, C) and (M_3, A). The expense matrix so obtained is given in Table.

Apply the usual Hungarian method of solving this problem. An optimal assignment so obtained is shown in Table.

	A	*B*	*C*	*D*	*E*
M_1	9	11	15	10	11
M_2	12	9	*M*	10	9
M_3	*M*	11	14	11	7
M_4	14	8	12	7	8
M_5	0	0	0	0	0

	A	*B*	*C*	*D*	*E*
M_1	$\boxed{0}$	2	6	1	2
M_2	3	$\boxed{0}$	*M*	1	$\cancel{0}$
M_3	*M*	4	7	4	$\boxed{0}$
M_4	7	1	5	$\boxed{0}$	1
M_5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

The total minimum expense (Rs) and optimal assignments made are as follows:

Machine	Location	Expense (in Rs 100)
M_1	A	9
M_2	B	9
M_3	E	7
M_4	D	7
M_5 (dummy)	C	0
		Total Rs 32

3. Jet Ltd. has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of 10. The higher the number, the greater is the preference. Certain of these flights are unsuitable to some pilots owing to domestic reasons. These have been marked with a x.

Flight Number

Pilot	1	2	3	4	5
A	8	2	×	5	4
B	10	9	2	8	4
C	5	4	9	6	×
D	3	6	2	8	7
E	5	6	10	4	3

What should be the allocation of the pilots to flights in order to meet as many preferences as possible?

Solution Since the problem is to maximize the total preference score, in order to apply the Hungarian method to solve this assignment problem, the equivalent expense matrix is required. This is obtained by subtracting all the elements of the given matrix from the largest element (= 10) including itself as shown in Table.

Equivalent Expense Matrix

	1	2	3	4	5
A	2	8	M	5	6
B	0	1	8	2	6
C	5	6	1	4	M
D	7	4	8	2	3
E	5	4	0	6	7

Perform the Hungarian method on Table to make assignments as shown in Table.

Opportunity Expense Table

	1	2	3	4	5
A	$\boxed{0}$	5	M	3	3
B	$\cancel{0}$	$\boxed{0}$	8	2	5
C	4	4	$\boxed{0}$	3	M
D	5	1	6	$\boxed{0}$	$\cancel{0}$
E	5	3	$\cancel{0}$	6	6

The solution shown in Table is not the optimal solution because there is no assignment in row E. Draw minimum number of lines to cover all zeros in the table and then subtract the smallest element (= 3) from all uncovered elements including itself and add it to the element at the intersection of two lines. The new table so obtained is shown in Table.

Repeat the Hungarian method to make assignments in Table. Since the number of assignments in Table is equal to the number of rows or columns, this solution is the optimal solution. The optimal assignment is as follows:

Pilot	Flight Number .	Preference Score
A	1	8
B	2	9
C	4	6
D	5	7
E	3	10
		Total 40

4.: Obtain a feasible solution of the following transportation problem by North-West corner rule for Digvijay Ltd.

Destinations

Origins		P	Q	R	S	Supply
	A	1	5	2	6	13
	B	9	10	3	8	17
	C	5	4	7	3	5
Requirement		5	11	15	4	35

The expense matrix shows the transportation expense in Rs. per unit.

Solution : First prepare a table ignoring the expense matrix.:

Destinations

Origins		P	Q	R	S	Supply
	A					13
	B					17
	C					5
Requirement		5	11	15	4	35

First select (1,1) cell. The supply of first row is 13 units and the requirement of first column is 5 units, therefore allocate 5 units to (1,1) cell. Hence the requirement of the first column will be satisfied and 13-5 = 8 units will remain to be allocated in the first row.

Now select (1,2) cell and allocate 8 units which is less among 8 and 11, hence the supply of first row will be exhausted and the requirement of second column of 11 - 8 = 3 units remain. Now allocate 3 units to (2, 2) cell so the requirement of second column will be satisfied and there will be surplus supply of 17 — 3 = 14 units in the second row. Now allocate 14 units which is less among 14 and 15 to (2,3) cell, so the second row will be satisfied and 15-4=1 unit is still required in the third column. Now allocate 1 unit which is less among 5 and 1 to (3,3) cell. So the third column will be satisfied and there will be 5 - 1 =4 unit surplus in the third row which is to be allocated to (3, 4) cell. Thus the feasible solution obtained is as follows.

Destinations

		P	Q	R	S	Supply
	A	$\boxed{5}$	$\boxed{8}$			~~13~~ ~~8~~
Origins	B		$\boxed{3}$	$\boxed{14}$		~~17~~ ~~14~~
	C			$\boxed{1}$	$\boxed{4}$	~~5~~ ~~4~~
Requirement		~~5~~	~~11~~ ~~3~~	~~15~~ ~~1~~	~~4~~	35

$x_1 = 5, x_2 = 8, x_{22} = 3, x_{24} = 14, x_{33} = 1, x_{34} = 4$

The total transportation expense

$Z = \sum c_{ij}x_{ij}$

$= 5(7) + 8(5) + 3(10) + 14(3) + 1(7) + 4(3)$

$= 35+40 + 30 + 42 + 7+12 = 166$ Rs.

5: Obtain a basic feasible solution of the following transportation problem by North-West corner rule for Haresh Ltd.

Destinations

Origins		D_1	D_2	D_3	D_4	D^5	Supply
	O_1	3	4	6	8	9	20
	O_2	2	0	1	5	8	30
	O_3	7	11	20	40	3	15
	O_4	2	1	9	14	16	13
Demand		40	6	8	18	6	78

Solution : Let us prepare the following table, ignoring the expense matrix :

Destinations

Origins		D_1	D_2	D_3	D_4	D_5	Supply
	O_1						20
	O_2						30
	O_3						15
	O_4						13
Demand		40	6	8	18	6	78

First of all select (1,1) cell and allocate 20 units which is less among 20 and 40, in (1,1) cell. Therefore the supply of the first row will be exhausted and there will remain a demand of 40 - 20 = 20 units in the first column. Now allocate 20 units which is less among 30 and 20 in (2,1) cell. So the demand of the first column will be satisfied and there will be surplus supply of 30 - 20 = 10 units in the second row.

Now select (2,2) cell and allocate 6 units to it which is less among 10 and 6, so the demand of the second column will be satisfied and there will remain an excess supply of 10 - 6 = 4 units in the second row. Now allocate 4 units which is less among 4 and 8 to (2,3) cell, so that the second row will be exhausted and 8-4=4 units will still be required in the third column. Now allocate 4 units which is less among 4 and 15 to (3, 3) cell, so that third column will be satisfied and there will be surplus supply of 15-4= 11 units in the third row. Then allocate 11 units which is less among 18 and 11 to (3,4) cell, so that third row will be exhausted and there will be demand of 18- 11 =7 units in the fourth column. Now allocate 7 units which is less among 13 and 7 to (4,4) cell, so that the fourth column will be satisfied. At the end allocate remaining 6 units to (4,5) cell. Thus supply of each row and demand of each column will be satisfied and the basic feasible solution can be obtained as follows :

Destinations

Origins	D_1	D_2	D_3	D_4	D_5	Supply
O_1	20					~~20~~
O_2	20	6	4			~~30~~ ~~10~~ ~~4~~
O_3			4	11		~~15~~ ~~11~~
O_4				7	6	~~13~~ ~~6~~
Demand	~~40~~ ~~20~~	~~6~~	~~8~~ ~~4~~	~~18~~ ~~7~~	~~6~~	78

6 Amdvad Ltd. employs typists on hourly piece-rate basis for their daily work. There are five typists and their charges and speed are different. According to an earlier understanding only one job is given to one typist and the typist is paid for a full hour even if he works for a fraction of an hour. Find the least expense allocation for the following data:

Typist	Rate per hour (Rs)	No. of Pages Typed/Hour
A	5	12
B	6	14
C	3	8
D	4	10
E	4	11

Job	No. of Pages
P	199

Q	175
R	145
S	298
T	178

Solution Applying Step 1 of the algorithm, we get Table in which entries represent the expense to be incurred due to assignment of jobs to various typists on a one-to-one basis.

	P	*Q*	*R*	*S*	*T*
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

	P	*Q*	*R*	*S*	*T*
A	2	2	$\cancel{0}$	4	0
B	6	4	$\boxed{0}$	10	2
C	0	1	$\cancel{0}$	1	2
D	2	4	$\cancel{0}$	4	2
E	2	0	$\cancel{0}$	0	2

Applying Step 2 of the algorithm we get the reduced opportunity expense matrix as shown in Table.

To determine assignments in Table we examine all the rows starting from A until a row containing only one zero element is located. Here rows B and D have only one zero element in the cells (B, R) and (D, R), respectively. Assignment is made in cells (B, R) first. All zeros in the assigned columns are now crossed off as shown in Table.

We now examine each column starting from column P. There is one zero in columns P, Q, S and T in the cells (C, P), (E, Q), (E, S) and (A, T). Assignment is made in these cells. All zeros are now either assigned or crossed off as shown in Table.

	P	*Q*	*R*	*S*	*T*
A	2	2	$\cancel{0}$	4	$\boxed{0}$
B	6	4	$\boxed{0}$	10	2
C	$\boxed{0}$	1	$\cancel{0}$	1	2
D	2	4	$\cancel{0}$	4	2
E	2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	2

	P	Q	R	S	T
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

The solution shown in Table is not optimal since only four assignments are made. Thus, to get the next best solution we go through the following steps.

(a) Mark () row D since it has no assignment.

(b) Mark () column R since row D has zero in this column.

(c) Mark () row B since column R has an assignment in row B.

(d) Since no other rows or columns can be marked, therefore, draw straight lines through the unmarked rows A, C and E and marked column R as shown in Table.

Develop the new revised expense table by selecting the smallest element among all uncovered elements by the lines in Table. Therefore, subtract this element from all uncovered elements including itself and add it to elements in the cells (A, R), (C, R) and (£, R), respectively which lies at the intersection of two lines. Another revised expense table so obtained is shown in Table.

Again repeat the procedure to find a new solution. The new assignment is shown in Table.

	P	Q	R	S	T
A	2	2	2	4	$\boxed{0}$
B	4	2	$\boxed{0}$	8	$\cancel{0}$
C	$\boxed{0}$	1	2	1	2
D	$\cancel{0}$	2	$\cancel{0}$	2	$\cancel{0}$
E	2	$\boxed{0}$	2	$\cancel{0}$	2

	P	Q	R	S	T
A	2	1	2	3	$\boxed{0}$
B	4	1	$\boxed{0}$	7	$\cancel{0}$
C	$\cancel{0}$	$\cancel{0}$	2	$\boxed{0}$	2
D	$\boxed{0}$	1	$\cancel{0}$	1	$\cancel{0}$
E	3	$\boxed{0}$	3	$\cancel{0}$	3

The solution shown in Table is also not optimal since only four assignments are made. Thus, to get the next best solution we follow Steps 6(a) to 6(d) of the algorithm to draw a minimum number of horizontal and vertical lines to cover all zero elements in Table. The new opportunity expense matrix obtained from Table by subtracting the smallest element (= 1) among all uncovered elements including itself by the lines and adding it to elements at the intersection of two lines as shown in Table.

The new solution obtained by repeating the procedure as explained earlier as shown in Table. Since both columns Q and S have two zeros, the arbitrary selection of a cell in any of these columns will give us an alternative solution also with the same total expense of assignment.

The pattern of assignments among typists and jobs, along with expense is as follows:

Typist	Job	Expense (in Rs)
A	T	75
B	R	66
C	S	114
D	P	80
E	Q	64
		Total Rs 399

7. Obtain a basic feasible solution of the following transportation problem by North-West corner rule for Rashmikant Ltd.

Destinations

Origins		D_1	D_2	D_3	D_4	D_5	Supply
	O_1	2	11	10	3	7	4
	O_2	1	4	7	2	1	8
	O_3	3	9	4	8	12	9
Requirement		3	3	4	5	6	21

Solution : First of all let us prepare the following table, ignoring the expense matrix.

Destinations

Origins		D_1	D_2	D_3	D_4	D_5	Supply
	O_1						4
	O_2						8
	O_3						9
Requirement		3	3	4	5	6	21

Begin with (1,1) cell and allocate 3 units to it which is less-among 4 and 3, so that the first column will be satisfied and there will be a surplus of 4-3 = 1 unit in the first row. Now select (1,2) cell and allocate 1 unit which is less among 1 and 3 in that cell so that the first row will be satisfied and in the second column there remains an excess demand of 3 - 1 = 2 units. Now in the cell (2,2) allocate 2 units which is less among 8 and 2 units so that the second column will be satisfied and there will be an excess of 8 - 2 = 6 units in the second row. Now allocate 4 units in the cell (2,3) which is less among 4 and 6 so that the third column will be satisfied and there will be 6 - 4 = 2 surplus units in the second row. Now allocate 2 units which is less among 2 and 5 in the cell (2,4) so that second row will be exhausted and there will be an excess demand of 5 - 2 = 3 units in the fourth column. Now allocate 3 units which is less among 3 and 9 to (3,4) cell so that the demand of the fourth column will be satisfied and there will be a surplus of 6 units in the third row which is to be allocated to (3,5) cell. Thus all rows and all columns are satisfied and the basic feasible solution is as follows :

Destinations

Origins	D_1	D_2	D_3	D_4	D_5	Supply
O_1	3	1				~~4~~ ~~1~~
O_2		2	4	2		~~8~~ ~~6~~ ~~2~~
O_3				3	6	~~9~~ ~~6~~
Requirement	~~3~~	~~3~~ ~~2~~	~~4~~	~~5~~ ~~3~~	~~6~~	21

Total expense $Z = \Sigma C_{ij} x_{ij}$

$= 3 \times 2 + 1 \times 11 + 2\times4 + 4\times7 + 2\times2$

$+ 3\times8 + 6\times12 = 153$ Rs.

8. Amit, a traveling salesman has to visit five cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The traveling expense (in Rs '000) of each city from a particular city is given below:

		A	B	To city C	D	E
	A	00	2	5	7	I
	B	6	00	3	8	2
From city	C	8	7	00	4	1
	D	12	4	6	00	5
	E	1	3	2	8	00

What is the sequence of visit of the salesman 'so that the expense is minimum?

Solution

Solving the given travelling salesman problem as an assignment problem by Hungarian method of assignment, an optimal solution is shown in Table 10.46. However, this solution is not the solution to the travelling salesman problem as it gives the sequence A — E — A. This violates the condition that salesmen can visit each city only once.

Optimal Solution

To city

From city	A	B	C	D	
A	∞	1	3	6	$\boxed{0}$
B	4	∞	$\boxed{0}$	6	~~0~~
C	4	3	∞	$\boxed{0}$	3
D	8	$\boxed{0}$	1	∞	1
E	$\boxed{0}$	2	~~0~~	7	∞

The 'next best' solution to the problem which also satisfies this extra condition of unbroken sequence of visit to all cities, can be obtained by bringing the next (non-zero) minimum element, i.e. 1 into the solution. In Table 10.46, the expense 1 occurs at three different places. Therefore, consider all three different cases one by one until the acceptable solution is reached.

Case 1 Make the unit assignment in the cell (A, B) instead of zero assignment in the cell (A, E) and delete row A and column B so as to eliminate the possibility of any other assignment in row A and column B. Now make the assignments in the usual manner. The resulting assignments are shown in Table.

To city

	A	B	C	D	E
A	~~∞~~	$\boxed{1}$	~~3~~	~~6~~	~~0~~
B	4	~~∞~~	$\boxed{0}$	6	~~0~~
C	4	~~3~~	∞	$\boxed{0}$	3
D	8	~~0~~	~~1~~	∞	$\boxed{1}$
E	$\boxed{0}$	~~2~~	~~0~~	7	∞

The solution given in Table gives the sequence: A->B, B->C, C->D, D->E, E->A. The expense corresponding to this feasible solution is Rs 15,000.

Case 2 If we make the assignment in the cell (D, C) instead of (D, E), then no feasible solution is obtained in terms of zeros or which may give expense less than Rs 15,000.

Hence the best solution is: A-B-C-D-E-A, and the total expense associated with this solution is Rs 15,000.

9. Niku Ltd. has a distribution depot in Greater Kailash Part I for Distributing ice cream in South Delhi. There are four vendors located in different parts of South Delhi (call them A,B, C and D) who have to be supplied ice cream every day. The following matrix displays the distances {in kilometres) between the depot and the four vendors:

What route should the company van follow so that the total distance travelled is minimized?

Solution Solving the given travelling salesman problem as an assignment problem by using the Hungarian method of assignment, we get the Table showing the optimal solution.

Optimal Solution

		To				
		Depot	*Vendor A*	*Vendor B*	*Vendor C*	*Vendor D*
	Depot	—	1.5	$\cancel{0}$	2	$\boxed{0}$
	Vendor A	1.5	—	0.5	$\boxed{0}$	0.5
From	*Vendor B*	$\boxed{0}$	0.5	—	1	$\cancel{0}$
	Vendor C	2	$\boxed{0}$	1	—	1.5
	Vendor D	$\cancel{0}$	0.5	$\boxed{0}$	1.5	—

A salesman has to visit five cities A, B, C, D and E. The distances (in hundred kilometres) between the five cities -are as follows:

		To city				
		A	B	C	D	E
	A	—	1	6	8	4
	B	7	—	8	5	6
From city	C	6	8	—	9	7
	D	8	5	9	—	8
	E	4	6	7	8	—

If the salesman starts from city A and has to come back to city A, which route should he select so that total distance travelled is minimum?

10.: A manager of Rajdut Ltd. has four subordinates and four different jobs. From past experience he knows the time (in minutes) each of the persons would take to complete different jobs. How should he assign different jobs to different persons so that the total time is minimum.

Jobs

Persons		P	Q	R	S
	A	12	15	18	8
	B	13	10	9	14
	C	10	12	15	13
	D	7	8-	9	14

Solution : First of all subtract the minimum element of each row from elements of that row. The matrix obtained will be as follows:

	P	Q	R	S
A	4	7	10	0
B	4	1	0	5
C	0	2	5	3
D	0	1	2	7

Now subtract the minimum element of each column from elements of that column. The resultant matrix will be as follows :

	P	Q	R	S
A	4	6	10	0
B	4	0	0	5
C	0	1	5	3
D	0	0	2	7

Now determine the least possible number of horizontal and vertical lines passing through all zeros. This is equal to 4 in this case, and it is equal to the number of rows (or columns). Therefore the solution is available at this stage.

First of all determine row having only one zero. This is observed in the cell (1,4). So give the first (*)assignment to this cell, and cross-off the first row and the fourth column. Now the third row also contains only one zero in the cell (3,1). Give an assignment (*) to this cell and cross of the third row and the first column. Now from the remaining elements the fourth row contains only one zero in the cell (4,2), so give an assignment (*) to the cell (4,2) and cross-off the fourth row and second column. Hence there is only one zero in the second row in the cell (2,3). So give an assignment to this cell. The solution obtained is as follows :

Jobs

Persons	P	Q	R	S
A				*
B			*	
C	*			
D		*		

Assign job S to person A

R to person B

P to person C

Q to person D

Total time required =8 + 9+10 + 8= 35 (minutes)

11: Maruti, a national car service has a surplus of one car in each of the cities A, B, C, D, E, F and a requirement of one car in each of the cities P, Q, R, S, T and U. The distance (in kms.) between cities with a surplus and cities with a requirement are given in the matrix below. How should the cars be dispatched so as to minimize the total distance travelled?

To

P		Q	R	S	T	U
A	41	62	39	52	25	51
B	22	29	49	65	81	50
C	27	29	60	51	32	32
D	45	50	48	52	37	43
E	29	40	39	26	30	33
F	82	40	40	60	51	30

Solution : First subtract minimum element of each row from each of the elements of that row. So the matrix obtained will be as follows:

p		Q	R	S	T	U
A	16	37	14	27	0	26
B	0	7	27	43	59	28
C	0	2	33	24	5	5
D	8	13	11	15	0	6
E	3	14	13	0	4	7
F	52	10	10	30	21	0

Now subtract minimum element of each column from all the elements of that column, so the matrix obtained will be as follows:

	P	Q	R	S	T	U
A	16	35	4	27	0	26
B	0	5	17	43	59	28
C	0	0	23	24	5	5
D	8	11	1	15	0	6
E	3	12	3	0	4	7
F	52	8	0	30	21	0

Now draw the minimum number of horizontal and vertical lines covering all zeros. Number of such lines is 5 which is less than the number of rows (or columns) = 6. Therefore the solution is not available at this stage. Now select the lowest element of the resultant matrix from the elements not covered by the horizontal or vertical lines. This element is 1. Therefore,

(i) Subtract 1 from each of the remaining elements not covered by horizontal or vertical lines.

(ii) Add 1 in each element of the intersection of horizontal and vertical lines.

Thus, the following matrix is obtained :

	P	Q	R	S	T	U
A	16	34	3	26	0	25
B	0	4	16	42	59	27
C	1	0	23	24	6	5
D	8	10	0	14	0	5
E	4	12	3	0	5	7
F	53	8	0	30	22	0

The least number of horizontal and vertical lines covering all zeros is 6 which is equal to the number of rows. Now select the row with only one zero. The first row contains only one zero in the cell (1, 5). So give the first assignment (*) to that cell and cross - off the first row and fifth column. Now fifth row contains only one zero in the cell (5, 4). So give an assignment (*) to the cell (5, 4) and cross - off the fifth row and the fourth column. Now the second row contains only one zero in the cell (2, 1). So give an assignment (*) to this cell and cross - off the second row and the first column.

Now the third row contains one zero in the cell (3,2). So give an asignment (*) to this cell and cross-off the third row and second column. Proceeding similarly give assignment to the cell (4, 3) and lastly to cell (6, 6). Therefore the solution obtained is as follows :

	P	Q	R	S	T	U
A					*	
B	*					
C		*				
D			*			
E				*		
F						*

i.e., Send cars form different cities as follows :

A $\rightarrow$T, B $\rightarrow$P, C $\rightarrow$Q, D $\rightarrow$R, E $\rightarrow$S, F $\rightarrow$U

Total minimum distance : = 25 + 22 + 29 + 48 + 26 + 30

= 180 (kms.)

12: Super Ltd. has 5 jobs to be done on 5 machines. The following matrix shows the return in Rs. of assigning i^{th} machine (i = 1, 2,.... 5) to the j^{th} job (j = 1, 2, ...5) Assign the five jobs to the five machines so as to maximise the total profit.

			Jobs		
Machines	a	b	c	d	e
1	12	18	20	8	20
2	20	4	8	1	16
3	21	7	13	10	17
4	2	18	21	16	16
5	9	13	20	15	19

Solution : Here we have to maximise the profit and hence we shall subtract each element of the matrix from the highest element 21 of the matrix. The following new matrix will be obtained :

	a	b	c	d	e
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

Now subtract the minimum element of each row from the elements of that row and obtain the following matrix :

	a	b	c	d	e
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

The minimum number of horizontal and vertical lines covering all zeros is 4 which is less than the number of rows (or columns), therefore the solution is not available at this stage. Now the lowest element among the elements uncovered by horizontal or vertical lines is 4. So we shall subtract 4 from each element uncovered by horizontal or vertical lines and shall add 4 to the elements which occur at the intersection of horizontal and vertical lines.

	a	b	c	d	e
1	12	0	0	7	0
2	0	10	8	10	0
3	0	8	4	2	0
4	23	1	0	0	5
5	15	5	0	0	1

Now the minimum number of horizontal and vertical lines passing through all zeros is 5 which is equal to the number of rows and hence the solution is obtained. There is only one zero in the second column and hence the workman b should be assigned the machine 1. Now cross - off the first row and second column. Now in the remaining rows and columns there are more than one zero, So we shall assign machine to workman arbitrarily. The second machine can be assigned to workman a. Cross - off the first column and second row. By doing so we shall get only one zero in the third row. So the third machine can be assigned to workman e. Similarly the fourth machine can be assigned to c and the fifth machine can be assigned to d. Thus, the final assignments will be as follows :

Workmen

Machines	a	b	c	d	e
1		*			
2	*				
3					*
4			*		
5				*	

Machine	Workman
1 →	b
2 →	a
3 →	e
4 →	c
5 →	d

Total profit = 18 + 20+17 + 21 + 15

= Rs. 91

13: The expense price of a machine is Rs. 5000 in Swastik Ltd. Its maintenance expense and the scrap value at the end of each year is given as follows. When should the machine be replaced ?

Year	1	2	3	4	5	6	7	8
Maintenance expense in Rs.	1500	1600	1800	2100	2500	2900	3400	4000
Scrap value in Rs.	3500	2500	1700	1200	800	500	500	500

Solution : We shall prepare the following table : Here C = Expense price = Rs. 5000

Year	Maintenance expense	Maintenance expense	Scrap value	Total expense	The average total annual
n	in Rs.	accumulated	in Rs.	= C - (3) + (2)	expense
	(1)	(2)	(3)	(4)	$T_A = (4)/n$
1	1500	1500	3500	3000	3000.00
2	1600	3100	2500	5600	2800.00
3	1800	4900	1700	8200	2733.33
4	2100	7000	1200	10800	2700.00*
5	2500	9500	800	13700	2740.00
6	2900	12400	500	16900	2816.67
7	3400	15800	500	20300	2900.00
8	4000	19800	500	24300	3037.50

The table indicates that the average total annual expense T_A is minimum during the fourth year. So it proves to be economical if the machine is replaced at the end of the fourth year.

14 : A machine expenses Rs. 12,200 and its scrap value is Rs. 200, a constant in Pratima Ltd. Its maintenance expense is known from the past experience as follows. After how many years should the machine be replaced ?

Year	1	2	3	4	5	6	7	8
Maintenance expense in Rs.	200	500	800	1200	1800	2500	3200	4000

Solution : Here the expense price = Rs. 12,200 and scrap value is fixed Rs. 200 for every year. We shall find the average total annual expense by preparing the following table :

Year n	Maintenance expense in Rs.	Accumulated maintenance expense in Rs.	Scrap value in Rs.	Total expense = C - (3) + (2)	The average total annual expense (in Rs.)
	(1)	(2)	(3)	(4)	$T_A = (4)/n$
1	200	200	200	12200	12200
2	500	700	200	12700	6350
3	800	1500	200	13500	4500
4	1200	2700	200	14700	3675
5	1800	4500	200	16500	3300
6	2500	7000	200	19000	3166.67*
7	3200	10200	200	22200	3171.43
8	4000	14200	200	26200	3275.00

The table indicates that the average total annual expense T_A is minimum during the sixth year. Therefore it is profitable to replace the machine at the end of sixth year.

15 : In Rahul Ltd., the purchase price of an item is Rs. 7000. Annual operating expense is Rs. 300 for the first year and then increases by Rs. 1500 every year. After how many years should the item be replaced ?

Solution : Here the expense price C of the item = Rs. 7000. As the scrap value is not mentioned we shall take it as zero.

We shall prepare the following table.

Year n	Operating expense in Rs.	Cumulative operating expense in Rs.	Scrap value in Rs.	Total expense in Rs.	The average total annual expense in Rs.
	(1)	(2)	(3)	(4)	$T_A = (4)/n$
1	300	300	-	7300	7300
2	1800	2100	-	9100	4550
3	3300	5400	-	12400	4133.33*
4	4800	10200	-	17200	4300.00
5	6300	16500	-	23500	4700

The table indicates that the average total annual expense T_A is minimum during the third year. Therefore it is desirable to replace the item at the end of the third year.

Practical Exercise:

1. In Rakesh Ltd., three jobs X, Y and Z are to be done on three machines P, Q, R. The following matrix shows the expenses of doing different jobs on different machines. Assign the three jobs to the three machines so as to minimize the total expense.

Machines (expense in Rs.)

Jobs	P	Q	R
X	21	25	31
Y	11	19	17
Z	15	17	13

2.. In Piyusha Ltd., a project work consists of four major jobs for which an equal number of contractors have submitted tenders. The tender amount quoted (in lakhs of rupees) is given in the matrix.

			Job		
		a	b	c	d
	1	10	24	30	15
Contractor	2	16	22	28	12
	3	13	20	32	10
	4	9	26	34	16

Find the assignment which minimizes the total expense of the project, when each contractor has to be assigned at least one job.

3. Solve the following assignment problem to minimize the total expense for Ravi Ltd.

Destinations

Origins	D_1	D_2	D_3	D_4	D_5
O_1	3	5	4	6	5
O_2	8	5	7	9	5
O_3	3	10	9	11	5
O_4	9	7	13	8	5
O_5	3	9	6	9	9

4. In Raghav Ltd., five workers are available to work with the machines and the respective expenses (in rupees) associated with each worker-machine assignment are given below. A sixth machine is available to replace one of the existing ones and the associated expenses are also given below.

		Machines					
		M1	M_2	M_3	M_4	M_5	M6
	W,	12	3-	6	—	5	9
	W_2	4	11	—	5	—	8
Workers	W3	8	2	10	9	7	5
	W4	—	7	8	6	12	10
	W_5	5	8	9	4	6	1

(a) Determine whether the new machine can be accepted.

(b) Also determine optimal assignment and the associated saving in expense.

5. Five works are to be assigned to five persons in Sarada Ltd. The following is the matrix showing the time (in hours) required for different persons to complete different works. Assign works to persons so that the total working hours is minimum.

Persons

Works	A	B	C	D	E
1	24	6	15	9	10
2	4	4	11	26	24
3	13	17	13	16	13
4	8	10	80	12	8
5	15	17	10	10	10

6. In Ravindra Ltd. five men are available to do five different jobs. From past records, the time (in hours) that each man takes to do each job is known and given in the following table:

Jobs

		1	II	III	IV	V
	A	2	9	2	1	1
	B	6	8	7	6	1
Men	C	4	6	5	3	1
	D	4	2	7	3	1
	E	5	3	9	5	1

Find the assignment of men to jobs that will minimize the total time taken.

7. Ajit Ltd. is producing a single product and is selling it through five agencies situated in different cities. All of a sudden, there is a demand for the product in another five cities not having any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to needy cities in such a way that the travelling distance is minimized. The distance between the surplus and deficit cities (in km) is given in the following table.

Deficit cities

	a	b	c	d	e
A	160	130	115	190	200
B	135	120	130	160	175
C	140	110	125	170	185
D	50	50	80	80	110
E	55	35	80	80	105

Determine the optimum assignment schedule.

8. Anish Ltd. has a surplus of one truck in each of the cities, 1, 2, 3, 4, 5 and 6; and a deficit of one truck in each of the cities 7, 8, 9, 10, 11 and 12. The distances (in km) between the cities with a surplus and cities with deficit are displayed in the table:

		8	9	10	11	12
1	31	62	29	42	15	41
2	12	19	39	55	71	40
3	17	29	50	41	22	22
4	35	40	38	42	27	33
5	19	30	29	16	20	23
6	72.	30	30	50	41	20

To

How should the trucks be displayed so as to minimize the total distance travelled?

9. Jet Ltd., picks up and delivers freight where customers want. The company has two types of aircraft. X and Y, with equal loading capacities but different operations expenses. These are shown below:

Type of Aircraft	Operating	Expenses (Rs)
	Empty	Loaded
X	1.00	2.00
Y	1.50	3.00

The present four locations of the aircrafts which the company has are; J - X: K - Y: L - Y, and M - X. Four customers of the company located at A, B. C and D want to transport nearly the same size of load to their final destinations. The final destinations are 600. 300, 1.000 and 500 km. from the loading points A, B, C and D, respectively.

Distances (in km) between the aircraft and the loading points are as follows:

Loading Point

	A	B	C	D
J	200	200	400	100
K	300	100	300	300
L	400	100	100	500
M	200	200	400	200

Aircraft Location

Determine the allocations which minimize the total expense of transportation.

10. Ankur Ltd. has requested bids for subcontracts on five different projects. Five companies have responded, their bids are represented below.

Bid Amounts ('000s Rs)

	1	II	III	IV	V
1	41	72	39 '	52	25
2	22	29	49	65	81
3	27	39	60	51	40
4	45	50	48	52	37
5	29	40	45	26	30

Determine the minimum expense assignment of subcontracts to bidders, assuming that each bidder can receive only one contract.

11. In Prakash Ltd., four persons are available to do four different jobs. From past records, the time (in hours) that each person takes to do each job is known, and is given in the following table:

	Job			
Person	I	II	III	IV
A	26	28	4	15
B	19	17	38	18
C	26	24	11	13
D	10	19	15	8

Find assignments of persons to jobs that will minimize the total time required.

12. Manit Ltd. is contemplating the introduction of three products 1, 2 and 3, in its three plants A, B and C. Only a single product is decided to be introduced in each of the plants. The unit expense of producing a product in a plant, is given in the following matrix.

		A	B	C
	1	8	12	—
Product	2	10	6	4
	3	7	6	6

(a) How should the product be assigned so that the total unit expense is minimized?

(b) If the quantity of different products to be produced is as follows, then what assignment shall minimize the aggregate production expense?

Product	Quantity (in units)
1	2,000
2	2,000
3	10,000

(c) What would your answer be if the three products were to be produced in equal quantities?

13. In Saptapadi Ltd., a department head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate of the time each man would take to perform each task is given in the effectiveness matrix below. How should the tasks be allocated, one to a man, so as to minimize the total man hours.

Man

Task		I	II	m	IV
	A	8	26	17	11
	B	13	28	14	26
	C	38	19	18	15
	D	19	26	24	10

14. Milap truck - rental service has a surplus of one truck in each of the cities A, B, C, D, E and F, and a deficit of one truck in each of the cities 1, 2, 3, 4, 5 and 6. The distance (in km.) between the cities with a surplus and cities with a deficit are displayed below. Find the best programme to minimize the total distance.

To

	1	2	3	4	5	6
A	26	57	24	10	36	37
B	7	14	34	66	35	50
C	12	24	45	17	17	36
D	30	35	33	22	28	37
E	14	25	24	15	18	11
F	67	25	25	36	15	25

15. The marketing director of Swaraj Ltd. is faced with a problem of assigning 5 senior managers to six zones. From past experience he knows that the efficiency percentage judged by sales, operating expenses, etc., depends on manager-zone combination. The efficiency of different managers is given below:

		Zones					
		I	II	III	IV	V	VI
	A	73	91	87	82	78	80
	B	81	85	69	76	74	85
Manager	C	75	72	83	84	78	91
	D	93	96	86	91	83	82
	E	90	91	79	89	69	76

16. In kash Ltd., the chairman of a store wants to assign 4 works W_1, W_2, W_3, W_4 to 4 persons P_1, P_2, P_3, P_4. Only one work is to be assigned to each person. The time to be taken by each person for different works is as follows. Assign the works in such a manner that the works are finished in minimum time.

Persons

Works	P_1	P_2	P_3	P_4
W_1	10	26	19	12
W_2	15	27	8	13
W_3	40	22	16	14
W_4	17	23	22	9

17. In Ram ltd., five persons are to be assigned five works. The following matrix shows the time each person will take to complete different works. Assign works to persons so that the total time is minimum.

Persons						
		I	II	III	IV	V
	A	15	25	50	75	40
	B	20	35	75	90	40
Works	C	40	60	100	100	60
	D	25	25	40	50	30
	E	50	50	75	125	50

18. In Suraj Ltd., a department have five employees and five jobs are to be performed. The time each man will take to perform each job is given in the matrix below. How should the jobs be allocated one per employee, so as to minimize the total man-hours ?

Employee

		I	II	III	IV	V
	A	10	5	13	15	16
Job	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

19. Solve the following assignment problem to maximize the total profit for Surya Ltd.

(Profit in Rs.)

	D_1	D_2	D_3	D_4
O_1	3	4	11	9
O_2	5	7	8	9
O_3	5	6	6	7
O_4	4	6	8	8

20. Rajiv, a salesman must travel from city to city to maintain his accounts. This week he has to leave his home base and visit other cities and return home. The table shows the distances (in km) between the various cities. The home city-is city A.

To city

		A	B	C	D	E
	A	—	375	600	150	190
	B	375	—	300	350	175
From city	C	600	300	—	350	500
	D	160	350	350	—	300
	E	190	175	500	300	—

Use the assignment method to determine the tour that will minimize the total distance of visiting all cities and returning home.

21. Amrit, a marketing manager has 5 salesmen and 5 sales districts. Considering the capabilities of the salesmen and the nature of the districts, the marketing manager estimates that sales per month in hundred rupees for each salesman in each district would be as follows. Find the best allocation to maximize the sales.

District

Salesman	1	2	3	4	5
A	42	48	50	38	50
B	50	34	38	31	46
C	51	37	43	40	47
D	52	48	51	46	46
E	39	43	50	45	49

22. Solve the following problem so as to maximise the profit for Kush Ltd.

(Profit in Rs.) Jobs

		A	B	C	D
	P	11	12	13	14
Workers	Q	14	15	16	17
	R	15	16	17	18
	S	18	17	16	15

23. Give assignment in the following problem for maximum profit for Laxman Ltd.

	D_1	D_2	D_3	D_4
O_1	2	3	4	5
O_2	5	6	7	8
O_3	6	7	8	9
O_4	9	8	7	6

24. Amjad, a factory owner finds from his past records that maintenance expense and resale price of a machine whose purchase price is Rs. 80,000 are as given below :

Year	1	2	3	4
Maintenance expense (in Rs.)	10,000	13,000	17,000	22,000
Resale price (in Rs.)	40,000	20,000	12,000	6,000
Year	5	6	7	8
Maintenance expense (in Rs.)	29,000	38,000	48,000	60,000
Resale price (in Rs.)	5,000	4,000	4,000	4,000

Determine at which time it is advisable to replace the machine.

25. Anup has purchased a refrigerator in Rs, 8000. From past records its maintenance expense and resale value during different years are as shown in the following table : Determine at which time it is advisable to replace it ?

Year	1	2	3	4	5	6
Maintenance expense (in Rs.)	1000 .	1200	1500	1800	2400	3000
Resale value (in Rs.)	5500	5000	4600	4000	3200	3000

26. A machine expenses Rs. 5,000 and its scrap value is Rs. 400. If the maintenance expense at the end of each year is as follows. determine the time at which it is advisable to replace it ?

Year	1	2	3	4	5	6	7	8
Maintenance expense in Rs.	1200	1500	1900	2500	3200	4000	4200	4400

27. A machine expenses Rs. 10,000. Annual operating expense is Rs. 400 for the firs year and then increases by Rs. 800 every year. After how many years should the machine be replaced ?

28. The expense of a machine is Rs. 16100 and its scrap value is only Rs. 1000. The maintenance expenses are found from past experience as follows :

Year	1	2	3	4	5	6	7	8
Maintenance expense (in Rs.)	200	350	500	1000	1400	2700	4000	5000

When should the machine be replaced ?

29. The purchase price of a machine is Rs. 9,000. Its maintenance expense for the first year is Rs. 200 and then it increases by Rs. 1,500 every year. Determine at which time it is profitable to replace the machine.

30. A machine expenses Rs. 6100 and its resale value is Rs. 100. Its maintenance expense is estimated as follows :

Year	1	2	3	4	5	6	7	8
Maintenance expense	100	250	400	600	900	1200	1600	2000

31. The price of a machine is Rs. 9000/- Its maintenance expense is Rs. 200/ - for the first year and then it increases by Rs. 2000/- per year. At what time is it profitable to replace the machine ?

32. Following table gives the running expense per year and resale price of a certain equipment whose purchase price is Rs. 5000. At what year is the replacement due ?

Year	1	2	3	4	5	6	7	8
Running Value (In Rs.)	1500	1600	1800	2100	2500	2900	3400	4000
Resale Value (In Rs.)	3500	2500	1700	1200	800	500	500	500

Unit : 4 : Transportation Problem

Introduction

The transportation problem is related mainly with the reducing the transportation cost and deciding the routes from the origin to destinations so that the cost of transportation is minimized. It can be treated as another important application of linear programming in the area of physical distribution (transportation) of goods and services from several supply origins to several demand destinations gets satisfied by this solution. Here the emphasis is on reducing the overall transportation cost and increasing the overall efficiencies. For solving such types of problems, transportation algorithms, namely the Stepping Stone Method and the MODI (modified distribution) Method, have been developed for this purpose.

Here, in transportation problem the main objective is to reduce the time of reaching from origin to the destination as well as to reduce the cost related to transportation. Here, we have to see, the cost is minimized in every respect when the firm is facing the problem of transportation. The transportation algorithm discussed in this chapter is applied to minimize the total expense of transporting a homogeneous commodity (product) from supply origins to demand destinations. However, it can also be applied to the maximization of some total value or utility, for example, financial resources are distributed in such a way that the profitable return is maximized.

The transportation problem has some of its special features and properties which we shall discuss here.

Feasibility As long as the supply equals demand, there exists a feasible solution to the problem.

Integrality If the supplies and demands are integer, every basic solution integer values. Therefore, it is not necessary to resort to integer programming to find integer solutions.

Structure of the chapter:

4.1 Objectives:

By the end of this chapter the student will learn to know about

4.2 Mathematical Model Of Transportation Problem

The transportation problem is related with the minimization of the cost and time and efforts involved by selecting the best route of transportation. For solving the problem, it will be converted in the mathematical form and then after the solution will be found out.

The Transportation Method

The solution algorithm to a transportation problem may be summarized into the following steps:

Step 1 Formulate the problem and set up in the matrix form

A) The formulation of the transportation problem is similar to the LP problem formulation.

B) Here the objective function is the total transportation expense and the constraints are the supply and demand available at each source and destination, respectively.

Step 2 Obtain an initial basic feasible solution

In this chapter, we shall discuss following three different methods to obtain an initial solution:

(i) North-West Corner Method

(ii) Least Expense Method

(iii) Vogel's Approximation Method.

The initial solution obtained by any of the three methods must satisfy the following conditions:

(a) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints.

(b) The number of positive allocations must be equal to m + n - 1, when m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called non-degenerate basic feasible solution, otherwise, degenerate solution.

Step 3 Test the initial solution for optimality

(a) In this chapter, we shall discuss the Modified Distribution (MODI) method to test the optimality of the solution obtained in Step 2.

(b) If the current solution is optimal, then stop, otherwise determine a new improved solution.

Step 4 Updating the solution

Repeat Step 3 until an optimal solution is reached.

Methods For Finding Initial Solution

There are several methods available to obtain an initial basic feasible solution. Here we shall discuss only three different methods:

North-West Corner Method

Step 1. Starting with the cell at the upper left (north-west) corner of the transportation matrix, we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, i.e.,

X11 = min. (a1, b1).

Step 2. If bi > ai, we move down vertically to the second row and make the second allocation of magnitude x_{21} = min. (a2,b1-x11 in the cell (2, 1).

If b1 < a1, we move right horizontally to the second column and make the second allocation of magnitude x_{12} = min. (a1 — x11,, b2) in the cell (1, 2).

If bi = ai, there is a tie for the second allocation. One can make the second allocation of magnitude

x_{12} = min. (a1 -a1, b1) = 0 in the cell (1, 2)

or x_{21} = min. (A2, b1 – b1) = 0 in the cell (2, 1).

Step 3. Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

Least Cost Method

Step 1. Determine the smallest cost in the cost matrix of the transportation table. Let it be cij. Allocate xij = min. (ai, bi,) in the ceil (i. j).

Step 2. If xij = ai cross off the tth row of the transportation table and decrease bj by Of. Go to step 3.

If xij = bj. cross off the jth column of the transportation table and decrease ai by bj. Go to step 3.

If xij = ai = bj cross off either the tth row or the jth column but not both.

Step 3. Repeat steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Vogel's Approximation Method

Step 1. Calculate penalties by taking differences between the minimum and next to minimum unit transportation costs in each row and each column."

Step 2, Circle the largest Row Difference or Column Difference. In the event of a tie; choose either.

Step 3. Allocate as much as possible In the lowest cost cell of the row (or column) having a circled Row (or Column) Difference.

Step 4. In case the allocation is made fully to a row (or column), ignore that row (or column) for further consideration, by crossing it.

Step 5. Revise the differences again and cross out the earlier figures. Go to step 2.

Step 6. Continue the procedure until all rows and columns have been crossed out, ie., distribution is complete.

Transportation Algorithm (Modi Method)

Various steps involved in solving any transportation problem may be summarized in the following iterative procedure :

Step 1. Find the initial basic feasible solution by using any of the three methods discussed above.

Step 2. Check the number of occupied cells. If these are less than m + n - 1, there exists degeneracy and we Introduce a very small positive assignment of e (= 0) in suitable independent positions, so that the number of occupied cells is exactly equal to m + n- 1.

Step 3. For each occupied cell in the current solution, solve the system of equations

$u_i + V_j = c_{ij}$

starting initially with some $u_i = 0$ or $Vj = 0$ and entering successively the values of u_t and Vj in the transportation table margins.

Step 5. Examine the sign of each zij - Cij. If all Zij - Cij < 0, then the current basic feasible solution is an optimum one. If at least one Zij - Cij > 0, select the unoccupied cell, having the largest positive net evaluation to enter the basis.

Step 6. Let the unoccupied cell (r, s) enter the basis. Allocate an unknown quantity, say 0, to the cell (r, s). Identify a loop that starts and ends at the cell (r, s) and connects some of the basic cells. Add and subtract interchangeably, 6 to and from the transition cells of the loop in such a way that the rim requirements remain satisfied.

Step 7. Assign a maximum value to 0 in such a way that the value of one basic variable becomes zero and the other basic variables remain non-negative. The basic cell whose allocation has been reduced to zero, leaves the basis.

Step 8. Return to step 3 and repeat the process until optimum basic feasible solution has been obtained.

Practical Exercise:

1. Aditya, a dairy farm has three plants located in a state. The daily milk production at each plant is as follows:

Plant 1: 6 million liters,

Plant 2: 1 million liters, and

Plant 3: 10 million liters

Each day, the firm must fulfill the needs of its four distribution centers. Minimum requirement at each center is as follows:

Distribution center 1: 7 million liters,

Distribution center 2: 5 million liters,

Distribution center 3: 3 million liters, and

Distribution center 4: 2 million liters

Expense in hundreds of rupees of shipping one million liter from each plant to each distribution center is given in the following table:

Distribution Center

		D_1	D_2	D_3	D_4
	P_1	2	3	11	7
Plant	P_2	1	0	6	1
	P_3	5	8	15	9

Find initial basic feasible solution for given problem by using

(a) North-west corner rule

(b) Least expense method

(c) Vogel's approximation method

if the object is to minimize the total transportation expense.

Solution (a) North-West Corner Rule

Table

Distribution Center

		D_1	D_2	D_3	D_4	Supply
	P_1	2 (6)	3	11	7	$6 = a_1$
Plant	P_2	1 (1)	0	6	1	$1 = a_2$
	P_3	5	8 (5)	15 (3)	9 (2)	$10 = a_3$
Demand		$7 = b_1$	$5 = b_2$	$3 = b_3$	$2 = b_4$	

(i) Comparing a_1 and b_1, since $a_1 < b_1$: allocate $x_{11} = 6$. This exhausts the supply at P_1 and leaves 1 unit as unsatisfied demand at D_1.

(ii) Move to cell (P_2, D_1). Compare a_2 and b_1 (i.e. 1 and 1). Since $a_2 = b_1$, allocate $x_{21} = 1$.

(iii) Move to cell (P_3, D_2). Since supply at P_3, is equal to the demand at D_2, D_3 and D_4, therefore, allocate $x_{32} = 5$, $x_{33} = 3$ and $x_{34} = 2$.

It may be noted that the number of allocated cells (also called basic cells) are 5 which is one less than the required number $m + n - 1$ $(3 + 4 - 1 = 6)$. Thus, this solution is the degenerate solution. The transportation expense associated with this solution is:

Total expense = Rs $(2 \times 6 + 1 \times 1 + 8 \times 5 + 15 \times 3 + 9 \times 2) \times 100$ = Rs 11,600

(b) Least Expense Method

Table

Distribution Center

		D_1	D_2	D_3	D_4	Supply
	P_1	2 (6)	3	11	7	6
Plant	P_2	1	0 (1)	6	1	1
	P_3	5 (1)	8 (4)	15 (3)	9 (2)	10
Demand		7	5	3	2	

(i) The lowest unit expense in Table is 0 in cell (P_2, D_2), therefore maximum possible allocation which can be made here is 1. This exhausts the supply at plant P_2, therefore, row 2 is crossed out.

(ii) The next, lowest unit expense is 2 in cell (P_1, D_1). The maximum possible allocation which can be made here is 6. This exhausts the supply at plant P_1, therefore, row P_1 is crossed out.

(iii) Since the total supply at plant P_3 is now equal to the unsatisfied demand at all the four distribution centers, therefore, maximum possible allocations satisfying the supply and demand conditions are made in cells (P_3, D_1), (P_3, D_2), (P_3, D_3) and (P_3, D_4).

The number of allocated cells in this case are six which is equal to the required number m + n -1 (3+4-1=6). Thus, this solution is non-degenerate. The transportation expense associated with this solution is

Total expense = Rs $(2 \times 6 + 5 \times 1 + 8 \times 4 + 15 \times 3 + 9 \times 2) \times 100$ = Rs 11,200

(c) Vogel 's Approximation Method

The method is self-explanatory as shown in Table.

Table

Distribution Centre

		D_1	D_2	D_3	D_4	Supply	Row penalty		
	P_1	2 (1)	3 (5)	11	7	6	1	1	5
Plant	P_2	1	0	6	1 (1)	1	0	—	—
	P_3	5 (6)	8	15 (3)	9 (1)	10	3	3	4

Demand	7	5	3	2
Column penalty	1	3	5	6
	3	5	4	2
	3	—	4	2

The number of allocated cells in Table 9.8 are six, which is equal to the required number m + n - 1 (3 + 4 - 1 = 6), therefore, this solution is non-degenerate. The transportation expense associated with this solution is

Total expense = Rs $(2 \times 1 + 3 \times 5 + 1 \times 1 + 5 \times 6 + 15 \times 3 + 9 \times 1) \times 100$ = Rs 10,200,

It can be seen that the total transportation expense found by VAM is lower than the expenses of transportation determined by the other two methods. Therefore, it is of advantage to use this method in order to reduce computational time required to obtain optimum solution.

2. Axe Ltd. has received a contract to supply gravel for three new construction projects located in towns A, B and C. Construction engineers have estimated the required amounts of gravel which will be needed at these construction projects:

Project Location	Weekly Requirement (Truckloads)
A	72
B	102
C	41

The company has 3 gravel pits located in towns X, Y and Z. The gravel required by the construction projects can be supplied by three pits. The amount of gravel which can be supplied by each pit is as follows:

Plant	:	X	Y	Z
Amount available (truckloads)	:	76	82	77

The company has computed the delivery expense from each pit to each project site. These expenses (in Rs) are shown in the following table:

Project location

A	B	C
X 4	8	8
Y 16	24	16
Z 8	16	24

Schedule the shipment from each pit to each project in such a manner so as to minimize the total transportation expense within the constraints imposed by pit capacities and project requirements. Also find the minimum expense.

Solution The total plant availability of 235 truckloads exceeds the total requirement of 215 truckloads by 20 truckloads. The excess truckload capacity, 20 is handled by adding a dummy project location (column), D_{excess} with a requirement equal to 20. We use zero unit transportation expenses to the dummy project location. The modified transportation table is shown in Table.

Table: Initial Solution

	A	*B*	*C*	D_{excess}	*Supply*
W	4	8 (35)	8 (41)	0	76
X	16	24 (62)	16	0 (20)	82
Y	8 (72)	16 (5)	24	0	77
Demand	72	102	41	20	235

The initial solution is obtained by using Vogel's approximation method as shown in Table. It may be noted that 20 units are allocated from pit X to dummy project location D. This means pit X is short by 20 units.

Now in order to apply optimality test, calculate u_i's and v_j's corresponding to rows and columns respectively in the same way as discussed before. The values are shown in Table.

	A	*B*	*C*	D_{excess}	*Supply*	u_i
W	4 +4	8 (+) 35	8 41 (−)	0 +16	76	$u_1 = 8$
X	16	24 (−) 62	16 (+) −8	0 20	82	$u_2 = 24$
Y	8 72	16 5	24 +8	0 +8	77	$u_3 = 16$
Demand	72	102	41	20	235	
v_j	$v_1 = -8$	$v_2 = 0$	$v_3 = 0$	$v_4 = -24$		

In Table, all opportunity expenses d_{ij}'s are not positive, the current solution is not optimal. Thus, the unoccupied cell (X, C) where d_{23} = - 8 must enter into the basis and cell (W, C) must leave the basis as shown by closed path. The new solution is shown in Table.

Table

	A	*B*	*C*	D_{excess}	*Supply*	u_i
W	4 +4	8 76	8 +8	0 +16	76	$u_1 = -16$
X	16 0	24 21	16 41	0 20	82	$u_2 = 0$
Y	8 72	16 5	24 +16	0 +8	77	$u_3 = -8$
Demand	72	102	41	20	235	
v_j	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Since all opportunity expenses d_{ij}'s are non-negative in Table, the current solution is optimal. The total minimum transportation expense associated with this solution is:

Total expense = $8 \times 76 + 24 \times 21 + 16 \times 41 + 0 \times 20 + 8 \times 72 + 16 \times 5$ = Rs 2,424.

3. Rajpara Ltd. company has factories at F_1, F_2, and F_3, which supply to warehouses at W_1, W_2 and W_3 Weekly factory capacities are 200, 160 and 90 units, respectively. Weekly warehouse requirement are 180, 120 and 150 units, respectively. Unit shipping expenses (in rupees) are as follows:

Warehouse

Factory		W_1	W_2	W_3	Supply
	F_1	16	20	12	200
	F_2	14	8	18	160
	F_3	26	24	16	90
	Demand	180	120	150	450

Determine the optimal distribution for this company to minimize total shipping expense.

Solution Initial basic feasible solution obtained by North-West Corner Rule is given in Table.

Initial Solution

	W_1	W_2	W_3	Supply
F_1	16 (180)	20 (20)	12	200
F_2	14	8 (100)	18 (60)	160
F_3	26	24	16 (90)	90
Demand	180	120	150	450

The initial solution has m + n - 1 = 3+3 - 1 = 5 allocations. Therefore, it is a non-degenerate solution. The optimally, test can, therefore, be performed. The total transportation expense associated with this solution is

Total expense = 16 × 180 + 20 × 20 + 8 × 100 + 18 × 60 + 16 × 90 = Rs 6,800

Determining the values of u_i'S and v_j's as usual by assigning $u_1 = 0$ arbitrarily. Given $u_1 = 0$, the values of others variables so obtained by using the equation, $c_{ij} = u_i + v_j$ for occupied cells are shown in Table.

Table

	W_1	W_2	W_3	Supply	u_i
F_1	16 (180)	20 (−) (20)	12 (+) −18	200	$u_1 = 0$
F_2	14 +10	8 (+) (100)	18 (60) (−)	160	$u_2 = -12$
F_3	26 +24	24 +18	16 (90)	90	$u_3 = -14$
Demand	180	120	150	450	
v_j	$v_1 = 16$	$v_2 = 20$	$v_3 = 30$		

$c_{11} = u_1 + v_1$ or $16 = 0 + v_1$ or $v_1 = 16$

$c_{12} = M, + v_2$ or $20 = 0 + v_2$ or $v_2 = 20$

$c_{22} = u_2 + v_2$ or $8 = u_2 + 20$ or $u_2 = - 12$

$c_{23} = u_2 + v_3$ or $18 = -12 + v_3$ or $v_3 = 30$

$c_{33} = u_3 + v_3$ or $16 = u_3 + 30$ or $u_3 = -14$

The opportunity expense for each of the unoccupied cells is determined by using the equation, $d_{ij} = c_{ij} - (u_i + v_j)$ as shown below:

$d_{13} = c_{13} - (u_1 + v_3) = 12 - (0 + 30) = -18$

$d_{21} = c_{21} - (u_2 + v,) = 14 - (-12 + 16) = 10$

$d_{31} = c_{31} - (u_3 + v_1) = 26 - (-14 + 16) = 24$

$d_{32} = c_{32} - (u_3 + v_2) = 24 - (-14 + 20) = 18$

The value of $d_{13} = - 18$ in the cell (F_1, W_3) indicates that the total transportation expense can be reduced in a multiple of 18 by introducing this cell in the new transportation schedule. To see how many units of .the commodity could be allocated to this cell (route) we shall form a closed path as shown in Table.

The largest number of units of the commodity which should be allocated to the cell (F_1, W_3) is 20 units because it does not violate the supply and demand restrictions (minimum allocation among the occupied cells bearing negative sign at the corners of the loop). The new transportation schedule (solution) so obtained-is shown in Table.

Table

	W_1	W_2	W_3	Supply
F_1	16 (180)	20	12 (20)	200
F_2	14	8 (120)	18 (40)	160
F_3	26	24	16 (90)	90
Demand	180	120	150	450

The total transportation expense associated with this solution is

Total expense = 16 × 180 + 12 × 20 + 8 × 120 + 18 × 40 + 16 × 90 = Rs 6,240

To test the optimality of the new solution shown in Table, again calculate the opportunity expense of each unoccupied cell in the same manner as discussed earlier. The calculations for u_i's, vj's and d_{ij}'s are shown in Table.

	W_1	W_2	W_3	Supply	u_i
F_1	16 (-)(180)	20 18	12 (20)(+)	200	$u_1 = 12$
F_2	14 (+) -8	8 (120)	18 (40)(-)	160	$u_2 = 18$
F_3	26 6	24	16 (90)	90	$u_3 = 16$
Demand	180	120	150		
v_j	$v_1 = 4$	$v_2 = -10$	$v_3 = 0$		

$c_{13} = u_1 + v_3$ or $12 = u_1 + 0$ or $u_1 = 12$

$c_{23} = u_2 + v_3$ or $18 = u_2 + 0$ or $u_2 = 18$

$c_{33} = u3 + v3$ or $16 = u_3 + 0$ or $u3 = 16$

$c_{11} = u_1 + v_1$ or $16 = 12 + v_1$ or $v_1 = 4$

$c_{22} = u_2 + v_2$ or $8 = 18 + v_2$ or $v_2 = -10$

$d_{12} = c_{12}-(u_1 + v_2)$ or $20 - (12-10) = 18$

$d21 = c_{21} - (u_2 + v_1)$ or $14 - (18+ 4) = -8$

$d_{31} = c_{31} - (u_3 + v_1)$ or $26 - (16+4) = 6$

$d_{32} = c_{32} - (u_3 + v_2)$ or $24 - (16-10) = 18$

The value of $d_{21} = -8$ in the cell (F_2, W_1) indicates that the total expense of transportation can further be reduced in a multiple of 8 by introducing this cell in the new transportation schedule. The new solution is obtained in the same manner by introducing 40 units of the commodity in the cell (f_2, W_1) as indicated in Table. The new solution is shown in Table.

Table

	W_1	W_2	W_3	Supply
F_1	16 (140)	20	12 (60)	200
F_2	14 (40)	8 (120)	18	160
F_3	26	24	16 (90)	90
Demand	180	120	150	

The total transportation expense associated with this solution is

Total expense = $16 \times 140 + 12 \times 60 + 14 \times 40 + 8 \times 120 + 16 \times 90$ = Rs 5,920

To test the optimality of the new solution shown in Table again calculate the opportunity expense of each unoccupied cell in the same manner as discussed earlier. The calculations are shown in Table.

Table: Optimal Solution

	W_1	W_2	W_3	Supply	u_i
F_1	16 (140)	20 +10	12 (60)	200	$u_1 = 16$
F_2	14 (40)	8 (120)	18 +8	160	$u_2 = 14$
F_3	26 +6	24 +10	16 (90)	90	$u_3 = 20$
Demand	180	120	150		
v_j	$v_1 = 0$	$v_2 = -6$	$v_3 = -4$		

$d_{12} = c_{12} - (u_1 + v_2)$ or $20 - (16-6) = 10$

$d_{23} = c_{23} - (u_2 + v_3)$ or $18 - (14-4) = 8$

$d_{31} = c_{31} - (u_3 + v_1)$ or $26 - (20 + 0) = 6$

$d_{32} = c_{32} - (u_3 + v_2)$ or $24 - (20 - 6) = 10$

Since-none of the unoccupied cells in Table has a negative opportunity expense value, therefore, total transportation expense cannot be reduced further. Thus, the solution shown in Table is the optimal solution giving optimal transportation schedule with a total expense of Rs 5,920.

4 : Solve the following transportation problem for Urvish Ltd., by Vogel's method and find the total expense.

Destinations

Origins	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Requirement	20	40	30	10	100

Solution : First of all we shall obtain the differences of the least expense and next higher expense in each row and in each column of the expense matrix.

	D_1	D_2	D_3	D_4	Supply	Difference d_1
O_1	1	2	1	4	30	0
O_2	3	3	2	1	50	1
O_3	4	2	5	9	20	2
Requirement	20	40	30	10		
	2	0	1	3		

Among the differences the maximum difference is 3 in the fourth column. In this column the minimum expense is 1 in (2,4) cell. Allocate 10 units (minimum of 10 and 50) in this cell. The fourth column will be satisfied and 50 - 10 = 40 units will remain to be allocated in the second row.

Hatching the fourth column, the new differences will be as follows :

	D_1	D_2	D_3	D_4	Supply	d_1	d_2
O_1	1	2	1	4	30	0	0
O_2	3	3	2	$\boxed{10}$ 1	~~50~~ 40	1	1
O_3	4	2	5	9	20	2	2
Requirement	20	40	30	~~10~~			
	2	0	1	3			
	2	0	1	-			

Among the new differences the maximum difference is 2 in two places i.e. in the third row and in the first column.

In the cell (1, 1) of the first column the expense is minimum, hence allocation of 20 units will be made in (1, 1) cell. The revised table will be as follows :

	D_1	D_2	D_3	D_4	Supply	d_1	d_2	d_3
O_1	[20] 1	2	1	4	~~30~~ 10	0	0	1
O_2	3	3	2	[10] 1	~~50~~ 40	1	1	1
O_3	4	2	5	9	20	2	2	3
Requirement	~~20~~	40	30	~~10~~				
	2	0	1	3				
	2	0	1	-				
	-	0	1	-				

The differences for the third time are obtained. Among the new differences the maximum difference is 3 in the third row and in this row the minimum expense

is 2 in the cell (3, 2). We shall allocate 20 units (minimum of 20 and 40) in this cell. The third row will be satisfied and the revised table will be as follows :

	D_1	D_2	D_3	D_4	Supply	d_1	d_2	d_3	d_4
O_1	[20] 1	2	1	4	~~30~~ 10	0	0	1	1
O_2	3	3	2	[10] 1	~~50~~ 40	1	1	1	1
O_3	4	[20] 2	5	9	~~20~~	2	2	3	-
Requirement	~~20~~	~~40~~ 20	30	~~10~~					
	2	0	1	3					
	2	0	1	-					
	-	0	1	-					
	-	1	1	-					

All the fourth differences are same. But the minimum expense is 1 in (1, 3) cell. So 10 units will be allocated to that cell.

	D_1	D_2	D_3	D_4	Supply
O_1	[20] 1	2	[10] 1	4	~~30~~ ~~10~~
O_2	3	3	2	[10] 1	~~50~~ 40
O_3	4	[20] 2	5	9	~~20~~
Requirement	~~20~~	~~40~~ 20	~~30~~ 20	~~10~~	

Lastly 20 units will be allocated in (2, 2) cell and 20 units will be allocated in (2, 3) cell. The final allocation is shown in the following table :

	D_1	D_2	D_3	D_4
O_1	[20] 1	2	[10] 1	4
O_2	3	[20] 3	[20] 2	[10] 1
O_3	4	[20] 2	5	9

The basic feasible solution will be as follows :

$x_{11} = 20; x_{13} = 10; x_{22} = 20; x_{23} = 20; x_{24} = 10; x_{32} = 20$

Total expense = $20 \times 1 + 10 \times 1 + 20 \times 3 + 20 \times 2 + 10 \times 1 + 20 \times 2$

$= 20 + 10 + 60 + 40 + 10 + 40$

$= 180.$

5. : Solve the following transportation problem by Vogel's method for Manish Ltd.

Destinations

Origins	1	2	3	4	Supply
1	21	16	25	13	11
2	17	18	14	23	13
3	32	27	18	41	19
Requirement	6	10	12	15	

Solution : First of all we shall find out the differences between the least expense and the next higher expense in each row and each column.

Destinations

Origins	1	2	3	4	Supply	d_1	d_2	d_3	d_4
1	21	16	25	$\boxed{11}$ 13	~~11~~	3	–	–	–
2	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23	~~13~~ ~~9~~ ~~3~~	3	3	3	4
3	32	$\boxed{7}$ 27	$\boxed{12}$ 18	41	19	9	9	9	9
Requirement	~~6~~	~~10~~ 7	12	~~15~~ ~~4~~					
d_1	4	2	4	10					
d_2	15	9	4	18					
d_3	15	9	4	–					
d_4	–	9	4						

Among all the differences the maximum difference is 10 in the fourth column. In this column the minimum expense is 13 in the cell (1,4). From 11 units of the first row and 15 units of the fourth column we shall allocate 11 units in (1,4) cell. So that the first row will be satisfied and 15 — 11 =4 units remain to be allocated in the fourth column.

Among the second differences the maximum difference is 18 in the fourth, column. In this column the minimum expense is in (2,4) cell. In this cell we shall allocate 4 units (min. of 13 and 4). The fourth column will be satisfied and 13-4 = 9 units remain to be allocated in the second row.

Now among the third differences maximum difference is 15 in the first column. The minimum expense is 17 in (2,1) cell in this column. In this cell we shall allocate 6 units (min. of 9 and 6). Thus first column will be satisfied and 3 units will remain to be allocated in the second row.

Now in the fourth differences the maximum difference is 9 in two places. We shall allocate 3 units in (2, 3) cell. The second row will be satisfied and 10-3 = 7 units remain to be allocated in the second column. Finally 7 units will be allocated in (3, 1) cell and 12 units will be allocated in (3, 2) cell. The table showing final allocation is given below :

Destinations

Origins	1	2	3	4
1	21	16	15	$\boxed{11}$ 13
2	$\boxed{6}$ 17	$\boxed{3}$ 18	14	$\boxed{4}$ 23
3	32	$\boxed{7}$ 27	$\boxed{12}$ 18	41

The basic feasible solution is

$x_{14} = 11$; $x_{21} = 6$; $x_{22} - 3$; $x_{24} = 4$; $x_{32} = 7$, $x_{33} = 12$ Total transportation expense

$= 11 \times 13 + 6 \times 17 + 3 \times 18 + 4 \times 23 + 7 \times 27 + 12 \times 18$

$= 143 + 102 + 54 + 92 + 189 + 216$

$= 796$ Rs.

6. Anup Ltd. wants to ship 22 loads of his product as shown below. The matrix gives the kilometers from sources of supply to the destinations.

Destination

Source		D_1	D_2	D_3	D_4	D_5	Supply
	S_1	5	8	6	6	3	8
	S_2	4	7	7	6	5	5
	S_3	8	4	6	6	4	9
	Demand	4	4	5	4	8	25 \ 22

Shipping expense is Rs 10 per load per km. What shipping schedule should be used to minimize total transportation expense?

Solution Since the total destination requirement of 25 units exceeds the total resource capacity of 22 by 3 units, the problem is unbalanced. The excess requirement is handled by adding a dummy plant, with a capacity equal to 3 units. We use zero unit transportation expense to the dummy plant. The modified transportation table is shown in Table.

Table: Initial Solution

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6 (5)	6	3 (3)	8
S_2	4 (4)	7	7	6 (1)	5	5
S_3	8	4 (4)	6	6	4 (5)	9
S_{excess}	0	0	0	0 (3)	0	3
Demand	4	4	5	4	8	25

The initial solution is obtained by using Vogel's approximation method as shown in Table. Since the solution includes 7 occupied cells, therefore, the initial solution is degenerate. In order to remove degeneracy we assign A to unoccupied cell (S_2, D_5) which has minimum expense among unoccupied cells as shown in Table.

Table

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	5 +3	8 +5	6 (–) 5	6 +2	3 3 (+)	8	$u_1 = 0$
S_2	4 4	7 +2	7 –1	6 1 (+)	5 Δ (–)	5	$u_2 = 2$
S_3	8 +5	4 4	6 –1	6 +1	4 5	9	$u_3 = 1$
S_{excess}	0 +2	0 +1	0 (+) –2	0 3 (–)	0 –7	3	$u_4 = -4$
Demand	4	4	5	4	8	25	
v_j	$v_1 = 2$	$v_2 = 3$	$v_3 = 6$	$v_4 = 4$	$v_5 = 3$		

Determine u_i and v_j for occupied cells as shown in Table. Since opportunity expense in the cell (S_{excess}, D_3) is largest negative, it must enter the basis and the cell (S_2, D_5) must-leave the basis. The new solution is shown in Table.

Table

	D_1	D_2	D_3	D_4	D_5	Supply	u_i
S_1	5 +1	8 +5	6 (–) 5	6 0	3 3 (+)	8	$u_1 = 0$
S_2	4 4	7 +4	7 +1	6 1	5 +2	5	$u_2 = 0$
S_3	8 +3	4 4	6 –1	6 (+) –1	4 5 (–)	9	$u_3 = 1$
S_{excess}	0 +2	0 +3	0 (+) Δ	0 3 (–)	0 +3	3	$u_4 = -6$
Demand	4	4	5	4	8	25	
v_j	$v_1 = 4$	$v_2 = 3$	$v_3 = 6$	$v_4 = 6$	$v_5 = 3$		

Repeat the procedure of testing optimality of the solution given in Table. The optimal solution is shown in Table.

Table

	D_1	D_2	D_3	D_4	D_5	Supply
S_1	5	8	6	6	3 (8)	8
S_2	4 (4)	7	7	6 (1)	5	5
S_3	8	4 (4)	6 (2)	6 (3)	4	9
S_{excess}	0	0	0 (3)	0	0	3
Demand	4	4	5	4	8	25

The minimum total transportation expense associated with this solution

7. Obtain a feasible solution of the transportation problem by matrix minima method for Alkesh Ltd.

Sales Centres

Godowns	X	Y	Z	W	Supply
A	8	9	6	3	18
B	6	11	5	10	20
C	3	8	7	9	18
Demand	15	16	12	13	56

Solution : In the given matrix the cell (1,4) and (3,1) both have lowest expense 3, But in the cell (3, 1) 15 units can be allocated while in the cell (1, 4) 13 units can be allocated. Hence the allocation of 15 units is made in the cell (1, 3). So that the first column will be satisfied and there are still 18 - 15 = 3 units surplus in the third row. Cross-off the first column. So the resultant matrix is as under :

Sales Centres

	X	Y	Z	W	Supply
A	8	9	6	3	18
B	6	11	5	10	20
C	15 3	8	7	9	~~18~~ 3
Demand	~~15~~	16	12	13	

Now the other lowest expense 3 is in the cell (1,4). Allocate 13 units (min. of 13 and 18) to this cell, So that the fourth column is satisfied and there are still 18 - 13 = 5 units surplus in the first row. Cross-off the fourth column so the resultant matrix is as under :

	X	Y	Z	W	Supply
A	8	9	6	$\boxed{13}$ 3	~~18~~ 5
B	6	11	5	10	20
C	$\boxed{15}$ 3	8	7	9	~~18~~ 3
Demand	~~15~~	16	12	~~13~~	

Now the cell (2,3) contains the next lowest expense 5. Allocate 12 units (min of 12 and 20) to this cell, so that third column is satisfied and there are still 20 - 12 = 8 units surplus in the second row, Cross-off the third column, so the resultant matrix is as under

	X	Y	Z	W	Supply
A	8	9	6	$\boxed{13}$ 3	~~18~~ 5
B	6	11	$\boxed{12}$ 5	10	~~20~~ 8
C	$\boxed{15}$ 3	8	7	9	~~18~~ 3
Demand	~~15~~	16	~~12~~	~~13~~	

Now the lowest expense among the remaining elements is 8 in the cell (3,2). Allocate 3 units (min of 3 and 16) to this cell so that third row is satisfied. Now allocate 5 units to cell (1,2) and 8 units to cell (2,2). The basic feasible solution obtained is as follows :

Sales Centres

Godowns	X	Y	Z	W	Supply
A	8	$\boxed{5}$ 9	6	$\boxed{13}$ 3	~~18~~ ~~5~~
B	6	$\boxed{8}$ 11	$\boxed{12}$ 5	10	~~20~~ ~~8~~
C	$\boxed{15}$ 3	$\boxed{3}$ 8	7	9	~~18~~ ~~3~~
Demand	~~15~~	~~16~~ ~~13~~ ~~8~~	~~12~~	~~13~~	

i.e. $x_{12} = 5, x_{14} = 3,$ $x_{22} = 8,$

$x_{23} = 12, x_{31} = 15,$ $x_{32} = 3$

Total transportation expense $= 5\times9 + 13\times3 + 8\times11 + 12\times5 + 15\times3 + 3\times8$

$= 45 + 39 + 88 + 60 + 45 + 24$

$= 301$

8. The following table shows all the necessary information on the availability of supply to each warehouse, the requirement of each market and unit transportation expense (in Rs) from each warehouse to each market for Rajesh Ltd.

Market

	P	Q	R	S	Supply
Warehouse A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Demand	7	12	17	9	45

The shipping clerk has worked out the following schedule from experience: 12 units from A to Q, 1 unit from A to R, 8 units from A to S, 15 units from B to R, 1 units from C to P and 1 unit from C to R.

(a) Check and see if the clerk has the optimal schedule.

(b) Find the optimal schedule and minimum total transport expense.

(c) If the clerk is approached by a carrier of route C to Q, who offers to reduce his rate in the hope of getting some business, by how much the rate should be reduced before the clerk will offer him the business.

Solution (a) The shipping schedule determined by the clerk based on his experience is shown in Table.

Table: Initial Solution

	P	Q	R	S	Supply
A	6	3 (12)	5 (1)	4 (9)	22
B	5	9	2 (15)	7	15
C	5 (7)	7	8 (1)	6	8
Demand	7	12	17	9	45

The total transportation expense associated with this solution is

Total expense = $3\times12 + 5\times1 + 4\times9 + 2\times15 + 5\times7 + 8\times1$ = Rs150

Since the number of occupied cells (i.e. 6) is equal to the required number of occupied cells (i.e. m + n - 1) in a feasible solution, therefore solution is non-generate feasible solution. Now, to test the optimality of the solution given in Table we evaluate each unoccupied cell in terms of the opportunity expense associated with it in the usual manner as shown in Table.

In Table, cell (C, S) has a negative opportunity expense (i.e. - 1). Thus, this solution is not the optimal solution and, therefore, the schedule prepared by the shipping clerk is not optimal.

(b) Forming a closed loop to introduce the cell (C, S) into the new transportation schedule as shown in Table, the new solution is shown in Table.

Table

	P	Q	R	S	Supply	u_i
A	6 +4	3 (12)	5 (+) (1)	4 (9) (-)	22	$u_1 = 0$
B	5	9 +9	2 (15)	7 +6	15	$u_2 = -3$
C	5 (7)	7 +1	8 (-) (1)	6 (+) -1	8	$u_3 = 3$
Demand	7	12	17	9	45	
v_j	$v_1 = 2$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$		

While testing the optimality of the improved solution shown in Table, we found that the opportunity expenses in all the unoccupied cells are positive. Thus the current solution is optimal and the optimal schedule is to transport 12 units from A to Q; 2 units from A to R; 8 units from A to S; 15 units from B to R; 1 units from C to P and 1 unit from C to S. The total minimum transportation expense associated with this solution is

Total, expense. = 3 ×12 + 5×2 + 4×8 + 2×15 + 5×7 + 6×1 = Rs149

Table

	P	Q	R	S	Supply	u_i
A	6 +3	3 (12)	5 (2)	4 (8)	22	$u_1 = 0$
B	5 +5	9 +9	2 (15)	7 +6	15	$u_2 = -3$
C	5 (7)	7 +2	8 +1	6 (1)	8	$u_3 = 2$
Demand	7	12	17	9	45	
v_j	$v_1 = 3$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$		

(c) The total transportation expense will increase by Rs 2 (opportunity expense) if one unit of commodity is transported from C to Q. This means that the rate of the carrier on, the route C to Q should be reduced by Rs 2, i.e. from Rs 7 to Rs 5 so as to get some business of one unit of commodity only.

In case all the 8 units available at C are shipped through the route (C, Q), then the solution presented in Table may be read as shown in Table.

Table

	P	Q	R	S	Supply
A	6 (7)	3 (4)	5 (2)	4 (9)	22
B	5	9	2 (15)	7	15
C	5	7 (8)	8	6	8
Demand	7	12	17	9	45

The total expense of transportation associated with this solution is

Total expense = $6\times7 + 3\times4 + 5\times2 + 4\times9 + 2\times15 + 7\times8$ = Rs 186.

Thus, the additional expense of Rs 37 (=186 - 149) should be reduced from the transportation expense of 8 units from C to Q. Hence transportation expense per unit from C to Q should be at the most 7 - (37/8) = Rs2.38

9: Obtain the solution of the following transportation problem by matrix minima method for Raghuvir Ltd.

Store

	A	B	C	a_i
F_1	10	9	8	8
F_2	10	7	10	7
F_3	11	9	7	9
F_4	12	14	10	4
b_j	10	10	8	28

Solution : In the given matrix the cells (2, 2) and (3, 3) have the lowest expense 7. In the cell (2, 2) 7 units (minimum of 7 and 10) can be allocated. While in the cell (3, 3) 8 units (minimum of 8 and 9 can be allocated. As more units can be allocated in (3, 3) cell it will be preferred for allocation. Allocating 8 units in (3, 3) cell cross-off the third column and the resultant matrix will be as under.

	A	B	C	a_i
F_1	10	9	8	8
F_2	10	7	10	7
F_3	11	9	[8] 7	~~9~~ 1
F_4	12	14	10	4
b_j	10	10	~~8~~	

The next allocation will be made in the cell (2, 2). In this cell allocating 7 units (minimum of 7 and 10), cross-off the second row and 10 - 7 = 3 units are required in the second column. The resultant matrix will be as under :

	A	B	C	a_i
F_1	10	9	8	8
F_2	10	[7] 7	10	~~7~~
F_3	11	9	[8] 7	~~9~~ 1
F_4	12	14	10	4
b_j	10	~~10~~ 3	~~8~~	

Now the minimum expense 9 is in two cells (1,2) and (3, 2). As more unit can be allocated to (1, 2) cell it is preferred. Allocating 3 units in (1, 2) cell, cross-off the second column and 8-3 = 5 units remain in the first row. The matrix will be as under :

	A	B	C	a_i
F_1	10	[3] 9	8	~~8~~ 5
F_2	10	[7] 7	10	~~7~~
F_3	11	9	[8] 7	~~9~~ 1
F_4	12	14	10	4
b_j	10	~~10~~ ~~3~~	~~8~~	

In the end only first column remains hence 5, 1 and 4 units can be allocated respectively to (1, 1), (3, 1) and (4, 1) cells. The final allocation will be as under:

	A	B	C	a_i
F_1	[5] 10	[3] 9	8	~~8~~ ~~5~~
F_2	10	[7] 7	10	~~7~~
F_3	[1] 11	9	[8] 7	~~9~~ ~~1~~
F_4	[4] 12	14	10	~~4~~
b_j	~~10~~	~~10~~ ~~3~~	~~8~~	

The solution is as under :

$x_{11} = 5, x_{12} = 3, x_{22} = 7, x_{31} = 1, x_{33} = 8, x_{41} = 4$

Total transportation expense

$$Z = \sum\sum C_{ij} X_{ij}$$

$= 5 \times 10 + 3 \times 9 + 7 \times 7 + 1 \times 11 + 8 \times 7 + 4 \times 12$

$= 50 + 27 + 49 + 11 + 56 + 48$

$= 241$

10. In Nishu Ltd., goods have to be transported from sources S_1, S_2 and S_3 to destinations D_1, D_2 and D_3. The transportation expense per unit, capacities of the sources and requirements of the destinations are given in the following table.

	D_1	D_2	D_3	Supply
S_1	8	5	6	120
S_2	15	10	12	80
S_3	3	9	10	80
Demand	150	80	50	

.Determine a transportation schedule so that expense is minimized.

Solution Using North-West Corner Method, the non-degenerate initial basic feasible solution is given in Table.

Table: Initial Solution

	D_1	D_2	D_3	Supply
S_1	8 (120)	5	6	120
S_2	15 (30)	10 (50)	12	80
S_3	3	9 (30)	10 (50)	80
Demand	150	80	50	280

To test the optimality of the solution given in Table, calculate u_i, v_j and d_{ij} as usual as shown in Table.

Table

	D_1	D_2	D_3	*Supply*	u_i
S_1	8 (120)	5 +2	6 +2	120	$u_1 = -7$
S_2	15 (–) (30)	10 (50) (+)	12 +1	80	$u_1 = 0$
S_3	3 (+) –11	9 (30) (–)	10 (50)	80	$u_1 = -1$
Demand	150	80	50	280	
v_j	$v_1 = 15$	$v_2 = 10$	$v_3 = 11$		

Since the unoccupied cell (S_3, D_1) has the largest negative opportunity expense of - 11, therefore, cell (S_3, D_1) is entered into the new solution mix. The closed path for (S_3, D_1) is shown in Table. The maximum allocation to (S_3, D_1) is 30. However, when this amount is allocated to (S_3, D_1) both cells (S_2, D_1) and (S_3, D_2) become unoccupied because these two have same allocations. Thus, the number of positive allocations became less than the required number, m + n – 1 = 3 + 3 – 1 = 5. Hence, this is a degenerate solution as shown in Table.

Table

	D_1	D_2	D_3	*Supply*
S_1	8 (120)	5	6	120
S_2	15	10 (80)	12	80
S_3	3 (30)	9 (Δ)	10 (50)	80
Demand	150	80	50	280

Table 9.39

	A	*B*	*C*	D_{excess}	*Supply*	u_i
W	4 +4	8 (76)	8 + 8	0 + 16	76	$u_1 = -16$
X	16 0	24 (21)	16 (41)	0 (20)	82	$u_2 = 0$
Y	8 (72)	16 (5)	24 + 16	0 +8	77	$u_3 = -8$
Demand	72	102	41	20	235	
v_j	$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	$v_4 = 0$		

Table 9.38

	D_1	D_2	D_3	*Supply*
S_1	8 (70)	5	6 (50)	120
S_2	15	10 (80)	12	80
S_3	3 (80)	9	10	80
Demand	150	80	50	280

To remove degeneracy a quantity A is assigned to one of the cells that has become unoccupied so that there are m + n - 1 occupied cells. Assign A to either (S_2, D_1) or (S_3, D_2) and proceed with the usual solution procedure. The optimal solution is given in Table.

11 : Obtain a basic feasible solution of the following transportation problem by matrix minima method for Hitu Ltd.

Destinations

Origins		D	E	F	Supply
	A	60	40	240	3
	B	100	65	180	5
	C	260	210	60	6
Requirement		6	4	4	14

Solution : In the expense matrix the minimum expense is in (1, 2) cell. Allocate 3 units (min. of 3 and 4) to this cell. The first row will be satisfied and 1 unit remains to be allocated in the second column. The next minimum expense is in the cells (1,1) and (3,3). But the first row is satisfied hence allocate 4 units (min of 4 and 6) in (3,3) cell. Thus the third column will be satisfied and 2 units will remain to be allocated in the 3rd row. Proceeding this way the final allocation will be as follows :

Destinations

Origins	D	E	F	Supply
A	60	[3] 40	240	$\cancel{3}$
B	[4] 100	[1] 65	180	$\cancel{5}$ $\cancel{4}$
C	[2] 260	210	[4] 60	$\cancel{6}$ $\cancel{2}$
Demand	$\cancel{6}$ $\cancel{2}$	$\cancel{4}$ $\cancel{1}$	$\cancel{4}$	14

The feasible Solution is as follows :

$x_{12} = 3, x_{21} = 4, x_{22} = 1, x_{31} = 2, x_{33} = 4$

Total expense $= 3 \times 40 + 4 \times 100 + 1 \times 65 + 2 \times 260 + 4 \times 60$

$= 120 + 400 + 65 + 520 + 240$

$= 1345$

12. : Obtain a basic feasible solution of the following transportation problem by Vogel's method for Digu Ltd.

Sales Depots

Godowns		x	y	z	Supply
	A	3	1	1	20
	B	2	9	12	30
	C	10	2	5	50
Requirement		35	15	50	

Solution : First of all we shall find the difference of the least expense and the next higher expense for each row and each column and represent them as shown below :

	x	*y*	*z*	Supply	Difference
A	3	7	1	20	2
B	[30] 2	9	12	~~30~~	7
C	10	2	5	50	3
Demand	~~35~~ 5	15	50	100	
	1	5	4		

Among all the differences the maximum difference is 7 in the second row. In this row the minimum expense is 2 in (2,1) cell. To this cell we shall allocate 30 units which is less among the supply of second row and the requirement of first column. Thus the second row will be satisfied and 35 - 30 = 5 units will remain to be allocated in the first column. The new difference of the first column will now be 7. The new differences are found out and among new differences the maximum difference is 7 in the first column, and the lowest expense among the expenses of the remaining cells in this column is 3 in (1,1) cell. To this cell we shall therefore allocate 5 units which is minimum of 5 and 20. Thus first column will be satisfied and 15 units will be required to be allocated in the first row.

	x	y	z	Supply	d_1	d_2
A	[5] 3	7	1	~~20~~ 15	2	2
B	[30] 2	9	12	~~30~~	7	–
C	10	2	5	50	3	3
Demand	~~35~~ ~~5~~	15	50			
d_1	1	5	4			
d_2	7	5	4			

Again among the new differences the maximum difference is 6 in the first row. The minimum expense among the remaining cells of the first row is in the cell (1,3)- In this cell we shall allocate 15 units (minimum of 15 and 50). Thus the first row will be satisfied and 35 units will remain to be allocated in the third column. The revised table will be as under :

	x	y	z	Supply	d_1	d_2	d_3
A	[5] 3	7	[15] 1	~~20~~ ~~15~~	2	2	6
B	[30] 2	9	12	~~30~~	7	–	–
C	10	2	5	50	3	3	3
Demand	~~35~~ ~~5~~	15	~~50~~ 35				
d_1	1	5	4				
d_2	7	5	4				
d_3	–	5	4				

Lastly we shall allocate 15 units in the cell (3, 2) and 35 units in the cell (3,3)- The basic feasible solution is given in the following table :

	x	y	z	Supply	d_1	d_2	d_3
A	[5] 3	7	[15] 1	~~20~~ ~~15~~	2	2	6
B	[30] 2	9	12	~~30~~	7	–	–
C	10	[15] 2	[35] 5	50	3	3	3
Demand	~~35~~ ~~5~~	15	~~50~~ 35				
d_1	1	5	4				
d_2	7	5	4				
d_3	–	5	4				

The solution is as under :

$x_{11} = 5$; $x_{13} = 15$; $x_{21} = 30$; x_{32} - 15; $x_{33} = 35$.

Total transportation expense = $5 \times 3 + 15 \times 1 + 30 \times 2 + 15 \times 2 + 35 \times 5$

$= 15 + 15 + 60 + 30 + 175$

$= 295$ Rs.

Exercise:

1. Bharat Ltd., consider four bases of operation 5; and three targets T_j The tons of bombs per aircraft from any base that can be delivered to any target are given in the following table:

		Target (T_j)		
		T_1	T_2	T_3
Base (B_i)	B_1	8	6	5
	B_2	6	6	6
	B_3	10	8	4
	B_4	8	6	4

The daily sortie capability of each of the four bases is 150 sorties per day. The daily requirement in sorties over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets explaining each step.

2. Anish Ltd. has four warehouses, a, b, c and d. It is required to deliver a product from these warehouses to three customers A, B and C. The warehouses have the following amounts in stock:

Warehouse	:	a	b	c	d
No. of units	:	15	16	12	13

and the customers' requirements are

Customer	:	A	B	C
No. of units	:	18	20	18

The table below shows the expenses of transporting one unit from warehouse to the customer.

Find the optimal transportation routes.

3. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCR, (b) LCM and (c) VAM for Raju Ltd.

		Destination				
		D_1	D_2	D_3	D_4	*Supply*
Source	S_1	21	16	15	3	11
	S_2	17	18	14	23	13
	S_3	32	27	18	41	19
	Demand	6	10	12	15	

4. Akil Ltd. is manufacturing a single product has three plants I, II and III. They have produced 60, 35 and 40 units, respectively during this month. The firm had made a

commitment to sell 22 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D and 30 units to customer E. Find the minimum possible transportation expense of shifting the manufactured product to the five customers. The net unit expense of transporting from the three plants to the five customers is given below:

Customers

		A	B	C	D	E
	I	4	1	3	4	4
Plants	II	2	3	2	2	3
	III	3	5	2	4	4

5. The following table of Super Ltd. gives the expense of transporting material from supply points A, B, C and D to demand points E, F, G, H and I.

To

		E	F	G	H	I
	A	8	10	12	17	15
From	B	15	13	18	11	9
	C	14	20	6	10	13
	D	13	19	7	5	12

The present allocation is as follows:

A to E 90; A to F 10; B to F 150; C to F 10; C to G 50; C to I 120; D to H 210; D to I 70.

(a) Check if this allocation is optimum. If not, find an optimum schedule.

(b) If in the above problem, the transportation expense from A to G is reduced to 10, what will be the new optimum schedule?

6. Adinath steel company has three open hearth furnaces and five rolling mills. Transportation expenses (rupees per quintal) for shipping steel from furnaces to rolling mills are shown in the following table:

	M_1	M_2	M_3	M_4	M_5	Supply
F_1	4	2	3	2	6	8
F_2	5	4	5	2	1	12
F_3	6	5	4	7	7	14
Demand	4	4	6	8	8	

What is the optimal shipping schedule?

7. Consider the following unbalanced transportation problem of Mahavir Ltd.

To

		I	//	III	Supply
	A	5	1	7	10
From	B	6	4	6	80
	C	3	2	5	15
Demand		75	20	50	

Since there is not enough supply, some of the demands at these destinations may ot be satisfied. Suppose there are penalty expenses for every unsatisfied demand unit which are given by 5, 3 and 2 for destinations I, II and III, respectively. Find the optimal solution.

8. Determine an initial basic feasible solution to the following transportation problem by using (a) the least expense method, and (b)Vogel's approximation method for Akash Ltd.

Destination

		D_1	D_2	D_3	D_4	*Supply*
Source	S_1	1	2	1	4	30
	S_2	3	3	2	1	50
	S_3	4	2	5	9	20
	Demand	20	40	30	10	

9. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCM, (b) LCM, and (c) VAM for Ravi Ltd.

Destination

		D_1	D_2	D_3	D_4	*Supply*
Source	*A*	11	13	17	14	250
	B	16	18	14	10	300
	C	21	24	13	10	400
	Demand	200	225	275	250	

10. Bhavik Ltd., produces a small component for an industrial product and distributes it to five wholesalers at a fixed delivered price of Rs 2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3,000, 3,000, 10,000, 5,000 and 4,000 units to wholesalers I, II, III, IV and V, respectively. The monthly production capacities are 5,000, 10,000 and 12,500 at plants 1, 2 and 3, respectively. Respective direct expenses of production of each unit are Re 1.00 Re 0.90, and Re 0.80 at plants W, X and Y. Transportation expenses of shipping a unit from a plant to a wholesaler are as follows.

Wholesaler

		I	II	III	IV	V
	W	0.05	0.07	0.10	0.25	0.15
Plant	X	0.08	0.06	0.09	0.12	0.14
	Y	0.10	0.09	0.08	0.10	0.15

Find how many components each plant supplies to each wholesaler in order to maximize its profit. 4. The ABC Tool Company has a sales force of 25 men who operate from three regional offices. The company produces four basic product lines of hand tools. Mr Jain, the sales manager, feels that 6 salesmen are needed to distribute product line 1, 10 to distribute product line II, 4 for product line III and 5 salesmen for product line IV. The expense (in Rs) per day of assigning salesmen from each of the offices for selling each of the product lines are as follows:

Product Lines

		I	//	III	IV
Regional Office	A	20	21	16	18
	B	17	28	14	16
	C	29	23	19	20

11. Obtain an optimum basic feasible solution to the following degenerate transportation problem for Rashmikant Ltd.

To

		A	B	C	Supply
	X	7	3	4	2
From	Y	2	1	3	3
	Z	3	4	6	5
Demand		4	1	5	

12. Determine an initial basic feasible solution to the following transportation problem by using the North-West corner rule, -where O_i and D_j represent ith origin and jth destination, respectively for Nikita Ltd.

Destination

		D_1	D_2	D_3	D_4	Supply
Source	O_1	6	4	1	5	14
	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
	Demand	6	10	15	4	

13. Derive basic feasible solution of the following transportation problem by North-West corner rule for Vanita Ltd.

Destinations

Origins		D_1	D_2	D_3	D_4	D_5	Supply
	O_1	5	4	7	6	3	5
	O_2	3	5	3	2	6	9
	O_3	3	4	7	5	5	15
Requirement		7	4	9	7	2	29

14. Obtain feasible solution of the following transportation problem by North-West corner rule for Savita Ltd.

Origin,		D_1	D_2	D_3	D_4	Supply
	O_1	6	4	1	5	14
	O_2	8	9	2	6	17
	O_3	4	3	6	2	5
Demand		6	10	16	4	36

15. Obtain basic feasible solution of the following transportation problem by N. W. comer rule for Kazi Ltd.

Depots

Godowns	A	B	C	D	E	F	Availability
I	9	12	9	6	9	10	5
II	7	3	7	7	5	5	6
III	6	5	9	11	3	11	2
IV	6	8	11	2	2	10	9
Requirement	4	4	6	2	4	2	22

16. Solve the following transportation problem by North-West comer rule for Harry Ltd.

To

From	I	II	III	Supply
1	7	12	9	16
2	8	10	6	10
3	10	9	12	12
Demand	8	11	19	38

17. Solve the following problem by North-West corner rule for Jagdish Ltd.

Warehouse

Plant	1	2	3	4	Supply
A	5	4	9	2	32
B	7	6	10	7	28
Requirement	18	16	14	12	60

18. Solve the following problem by North-West corner rule and find the total transportation expense for Ram Ltd.

	I	II	III	IV	Availability
A	5	1	3	3	34
B	3	3	5	4	15
C	6	4	4	3	12
D	4	1	4	2	19
Requirement	21	25	17	17	80

19. Obtain feasible solution by using North-West corner method for Padmavati Ltd.

Destinations

Origins	A	B	C	D	E	Supply
I	10	8	11	10	8	35
II	12	10	8	11	7	55
III	15	6	10	12	9	40
IV	17	12	9	10	6	10
V	11	13	10	10	5	10
Demand	50	45	25	15	15	150

20. Solve the following problem by matrix minima method for Gantakar Ltd.:

To

From	(1)	(2)	(3)	(4)	Supply
A	8	5	9	7	20
B	6	4	2	10	40
C	6	1	3	3	60
Demand	20	50	25	25	120

21. The following table gives the expense matrix for a transportation problem. Obtain basic feasible solution by matrix minima method for Mahavir Ltd.

Destinations

		(1)	(2)	(3)	(4)	(5)	(6)	Availability

	A	5	3	7	3	8	5	3
	B	5	6	12	5	7	11	4
Origins	C	2	8	3	4	8	2	2
	D	9	6	10	5	10	9	8
Requirement		3	3	6	2	1	2	17

22. Obtain feasible solution of the following problem by matrix minima method for Sambhav Ltd.

To

From	P	Q	R	Supply
(D	5	6	7	6
(2)	12	8	4	10
(3)	3	10	14	3
Requirement	t 10	4	5	19

22. Solve the following transportation problem by matrix minima method. Also obtain the total transportation expense for Nasir Ltd.

	D_1	D_2	D_3	D_4	
O_1	3	6	3	12	30
O_2	9	7	6	3	50
O_3	12	6	15	27	20
	20	40	30	10	100

23. Obtain basic feasible solution of the following problem by matrix minima method for Power Ltd.

Origins	D_1	D_2	D_3	D_4	Supply
O_1	5	6	8	10	10
O_2	10	8	6	4	15
O_3	2	5	7	9	25
Demand	15	10	10	15	50

24. Obtain basic feasible solution of the following problem by Vogel's method for Kaushik Ltd.

To

From		A	B	C	Supply
	I	150	130	330	1
	II	190	145	270	3
	III	350	300	150	5
Demand		4	2	3	9

25. Obtain basic feasible solution of the following problem by Vogel's method for Maya Ltd.

Destinations

Origins		1	2	3	4	Supply
	A	21	16	25	13	11
	B	17	18	14	23	13
	C	32	27	18	41	19
Requirement		6	10	12	15	43

26. Solve the following problem by Vogel's method and find the total transportation expense for Hansa Ltd.

Sale - Centre

Godown	S_1	S_2	S_3	S_4
A	8	9	6	3
B	6	11	5	10
C	3	8	7	9

Availability	Number of units	Sales-	Requirements
Godowns		Centres	
A	19	S_1	15
B	12	S_2	6
C	14	S_3	11
		S_4	13

27. Find basic feasible solution of the following problem by using Vogel's method for Bipin Ltd.

Destinations

Origins		A	B	C	D	E	F	Supply
	1	6	12	9	6	9	10	5
	II	7	3	7	7	5	5	6
	III	6	5	9	11	3	11	2
	IV	6	8	11	2	2	10	9
Demand		4	4	6	2	4	2	22

28. Solve the following problem by Vogel's method for Dhanlaxmi Ltd.:

To

From A		B	C Supply	
I	18	22	10	20
II	25	11	20	22
III	15	30	7	18
Requirement 16		21	23 60	

29. Obtain basic feasible solution of the following transportation problem by Vogel's approximation method for Nirav Ltd.

	W_1	W_2	W_3	W_4	Supply
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	49	8	70	20	18
Demand	5	8	7	14	34

30. Obtain basic feasible solution of the following problem by Vogel's method for Kartik Ltd.

Destinations

Origins	P	Q	R	S	T	U	Supply
A	12	11	13	13	10	15	50
B	13	12	12	14	13	10	40
C	13	15	14	12	14	11	60
D	14	12	11	11	10	12	31
Requirement	30	50	20	40	30	11	

31. Solve the following transportation problem by using the three methods you know for Rajit Ltd.

Destinations

Origins		D_1	D_2	D_3	D_4	Supply
	O_1	11	6	15	3	16
	O_2	7	8	4	13	18
	O_3	22	17	8	31	24
Demand		11	15	17	15	58

32. Find initial basic feasible solutions of the following transportation problem by using any two methods for Kanpur Ltd.

Destinations

Origins	A	B	C	D	E	Supply
P	5	7	6	8	9	20
Q	9	8	10	4	11	35
R	10	12	9	7	8	40
S	6	6	7	8	8	15
Demand	15	10	20	30	35	110

33. Solve the following transpotation problem by any two methods for Nagpur Ltd. All entries are unit expense.

73	40	9	79	20	8
62	93	96	8	13	7
96	65	80	50	55	9
57	58	29	12	87	3
56	23	87	18	12	5
6	8	10	4	4	